Chern Character in Twisted and Equivariant K-Theory

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In this talk all our spaces will be paracompact Hausdorff and our bundles complex. Recall that the usual Chern character is an isomorphism

$$ch : K^*(X, \mathbb{Q}) \xrightarrow{\cong} H^{even/odd}(X, \mathbb{Q})$$

defined by the splitting principle and the fact that $ch(L) = \exp c_1(L)$ for L a line bundle. The goal of this talk is to generalize this isomorphism to the case of twisted K-theory, and then to equivariant K-theory.

Twisted Chern Character

Let's first get an idea of what the twisted Chern character is supposed to be, and why one should expect it to exist. A twist for (rational) K-theory over a space is the same as a bundle with fiber the classifying spectrum for (rational) K-theory $(BU \times \mathbb{Z}) \otimes \mathbb{Q}$. The usual Chern character gives an isomorphism between this spectrum and a product of Eilenberg MacLane spectra $\prod_{n\geq 0} K(\mathbb{Q}, 2n)$. Therefore K-theory twists are the same as bundles with fiber $\prod_{n\geq 0} K(\mathbb{Q}, 2n)$, which is the same as a twist for ordinary cohomology. The corresponding isomorphism between spaces of sections is the twisted Chern character.

A bit more precisely, the K-theory twists we consider are given by $BU(1)$-principal bundles. We have an action of $BU(1)$ on $\prod_{n\geq 0} K(\mathbb{Q}, 2n)$ given by multiplication by the Chern character. The associated bundle is the corresponding twisted cohomology bundle. Concretely, given a cover U_i for X trivializing the twist, we have transition line bundles

$$U_i \cap U_j = U_{ij} \leftarrow L_{ij}$$

Let ω_{ij} be a cocycle representing $c_1(L_{ij})$. Then the twisted ordinary cochains are trivialized over U_i as $\prod_{n\geq 0} C^*(U_i, \mathbb{Q})(2n)$, with transitions being multiplication by $\exp \omega_{ij}$. Now, choose $\nu_i \in C^2(U_i, \mathbb{Q})$ such that $\omega_{ij} = \nu_i - \nu_j$. Then we can use $\exp \nu_i$ to conjugate $\prod C^*(U_i, \mathbb{Q})(2n)$ so that transitions become trivial, but the differential becomes $d - d\nu_i$. Our model for twisted differential cohomology will then be the $C^*(X, \mathbb{Q})[[\beta]]$ with the twisted differential $d - \beta\eta$, where β is a formal variable of degree -2 (Bott element) and η is a global 3-cocycle obtained by patching $d\nu_i$ (it turns out that η is a representative of the Dixmier-Douady class of the twist).

We shall now carry this out with a concrete model of cohomology. Observe that in general $(d - \beta\eta)^2 = \eta^2\beta^2$ which is only cohomologous to zero. So we should work in a commutative model. Moreover, in order to find ν_i as above, we need vanishing of cohomology of our sheaf of cochains. Although the construction may be carried over $\mathbb{Q}$, we will work over manifolds and use de Rham cochains, which are indeed commutative and don't have cohomology thanks to the existence of partitions of unity. Our first goal will then be to find a cochain representative for the Chern character in differential forms. This is what Chern-Weil Theory does for us.

Chern-Weil theory

Let M be a manifold and $E \to M$ be a hermitian vector bundle. Let ∇ be a connection on E. We can extend ∇ to operators $\nabla : \Omega^p(M) \otimes E \to \Omega^{p+1}(M) \otimes E$ satisfying the Leibnitz rule. One may check that ∇^2 is $\Omega^*(M)$-linear, and so it is given by multiplication by a section F_∇ of $\Omega^2(M, \operatorname{End} E)$ (here by $\operatorname{End} E$ we mean skew-hermitian endomorphisms of E), called the curvature of the connection. Observe that $\nabla F_\nabla = 0$ (this is the Bianchi identity).

Let $P \in (S\mathfrak{u}_n^*)^{U(n)}$ be a polynomial on skew-hermitian matrices invariant under conjugation. Then for any hermitian vector space V one gets a well defined polynomial $P : \operatorname{End} V \to \mathbb{R}$. In particular, PF_∇ is a well defined $2d$-form on M, where d is the degree of P.

Proposition. *i. PF_∇ is closed.*

ii. The cohomology class of PF_∇ is independent of ∇.

Proof. i. Let $x \in M$ and $e_1, \ldots, e_n$ be a basis for E around x such that $\nabla e_i(x) = 0$. Write $F_\nabla = F_i^j e^i \otimes e_j$ with $F_i^j \in \Omega^2(M, \mathbb{R})$. Then the Bianchi identity implies that $dF_i^j(x) = 0$ and therefore $dPF_\nabla(x) = 0$.

ii. Let ∇' be a different connection. Pullback E to a vector bundle on $M \times I$. We may define a connection on this bundle by $\nabla_t = (1-t)\nabla + t\nabla' + d_t$. This interpolates between ∇ and ∇'. Then if $i_0, i_1 : M \to M \times I$ are the inclusions at time 0 and 1, we have

$$[PF_\nabla] = i_0^*[PF_{\nabla_t}] = i_1^*[PF_{\nabla_t}] = [PF_{\nabla'}]$$

□

As a consequence of the above results, we see that $[PF_\nabla]$ gives a well defined characteristic class on n-dimensional bundles. In fact, what one has is a morphism

$$(S\mathfrak{u}_n^*)^{U(n)} \to H^*(BU(n), \mathbb{R})$$

called the Chern-Weil map.

Theorem. *The Chern-Weil map is an isomorphism*

Proof. We apply the splitting principle:

$$\begin{array}{ccc} (S\mathfrak{u}_1^{n*})^{S_n} & \longrightarrow & H^*(BU(1)^n, \mathbb{R})^{S_n} \\ \uparrow & & \uparrow \\ (S\mathfrak{u}_n^*)^{U(n)} & \longrightarrow & H^*(BU(n), \mathbb{R}) \end{array}$$

The left upwards map is easily seen to be an isomorphism, and the right upwards map is an isomorphism by the theory of Chern classes. Then one has to show that the map on top is an isomorphism. This follows if we show that the Chern-Weil map for $n = 1$ is an isomorphism. This is the map

$$S\mathfrak{u}_1^* \to H^*(BU(1), \mathbb{R})$$

Here $S\mathfrak{u}_1^*$ is a polynomial algebra generated by an element p such that $p(i) = 1$, and $H^*(BU(1), \mathbb{R})$ is a polynomial algebra generated by c_1. This map respects degree, and by looking for example at the case of TS^2 one may see that p is mapped to $-2\pi c_1$. It follows that it is an isomorphism, as we wanted. □

The theorem shows that all characteristic classes may be obtained via the Chern-Weil construction. In particular, tracing isomorphisms one may show that the Chern character is given by

$$ch(E) = \left[\operatorname{tr} \exp \frac{iF_\nabla}{2\pi}\right]$$

which is the de Rham cocycle expression we were looking for

Connections and Twisted K-Theory

We now recall the construction of twisted K-theory via bundle gerbes and prove some preliminary results which will be used to construct the twisted Chern character.

Let M be a manifold and $M_0 \to M$ be a submersion (for example, an open cover). Let $M_1 = M_0 \times_M M_0$, $M_2 = M_0 \times_M M_0 \times_M M_0$, etc. Then the spaces M_i form a simplicial manifold

$$\ldots M_2 \substack{\rightarrow\\\rightarrow\\\rightarrow} M_1 \rightrightarrows M_0$$

which should be thought of as a model for M. A bundle gerbe is a (Hermitian) line bundle $L \to M_1$ with a multiplication

$$L_{01} \otimes L_{12} = L_{02}$$

satisfying a certain compatibility condition on M_3 (here by L_{ij} we mean the pullback of L to M_2 via the appropriate face map). Equivalently, it is a line bundle on M_1 satisfying the cocycle condition

$$L_{01} \otimes L_{12} \otimes L_{20} = 1$$

where 1 is the trivial bundle.

Claim. *There exists a connection ∇ on L such that the multiplication map is flat.*

Proof. Take any connection ∇ on L. It induces a connection $\delta\nabla$ on $L_{01} \otimes L_{12} \otimes L_{20}$. This bundle also has the trivial connection d. Let $\Gamma = \delta\nabla - d \in \Omega^1(M_2, i\mathbb{R})$. Observe that the spaces $\Omega^1(M_i, i\mathbb{R})$ belong to a (Čech) complex

$$\Omega^1(M_0, i\mathbb{R}) \xrightarrow{\delta} \Omega^1(M_1, i\mathbb{R}) \xrightarrow{\delta} \Omega^1(M_2, i\mathbb{R}) \xrightarrow{\delta} \ldots$$

which is acyclic since it computes the cohomology of the sheaf $\Omega^1_M \otimes i\mathbb{R}$. Now, it may be seen that Γ is closed so one may write $\Gamma = \delta\alpha$ with $\alpha \in \Omega^1(M_1, i\mathbb{R})$. Then $\nabla - \alpha$ is a multiplicative connection. □

From now on fix ∇ multiplicative. Let $\omega \in \Omega^2(M_1, \mathbb{R})$ be its curvature normalized so that it represents the first Chern class. The fact that ∇ is multiplicative implies that $\delta\omega = 0$, and so one may find $\nu \in \Omega^2(M_0, \mathbb{R})$ such that $\omega = \delta\nu$. Then

$$\delta d\nu = d\delta\nu = d\omega = 0$$

and so $d\nu$ descends to a closed 3-form η on M. Its cohomology class is the Dixmier-Douady invariant of the gerbe. The twisted de Rham complex is then defined to be the complex

$$(\Omega^*(M)[[\beta]], d - \beta\eta\wedge)$$

where β is a formal variable of degree -2.

Construction of the twisted Chern character

We now give the explicit construction of the twisted Chern character in the de Rham model. We shall assume for simplicity that the Dixmier-Douady class is torsion. In that case one may represent any element of twisted K-theory by a bundle gerbe module: this is a (Hermitian) vector bundle $E \to M_0$ together with a multiplication $L \otimes E_0 = E_1$ satisfying some compatibility condition on M_2.

By using the same kind of arguments that we used before, one may take a connection ∇ on E such that the multiplication is flat. Let F be its curvature. Then $\omega \operatorname{Id} + F_0 = F_1$ and therefore $\nu_0 \operatorname{Id} + F_0 = \nu_1 \operatorname{Id} + F_1$. This means that $\nu \operatorname{Id} + F \in \Omega^2(M_0, \operatorname{End} E)$ descends to M (here it is worth noting that although E is not defined on M, the bundle $\operatorname{End} E$ does descend to a bundle on M). One may therefore give the following

Definition. *The twisted Chern character is* ${}^{\tau}ch(E) = [\operatorname{tr}\exp(\beta(\nu \operatorname{Id} + F))]$.

Observe that we have

$$(d - \beta\eta\wedge)\operatorname{tr}\exp(\beta(\nu \operatorname{Id} + F)) = \operatorname{tr}(\beta(d\nu \operatorname{Id} + \nabla F)\exp(\beta(\nu \operatorname{Id} + F))) - \beta\eta\wedge \operatorname{tr}\exp(\nu \operatorname{Id} + F) = 0$$

and so the twisted Chern character defines a class in twisted cohomology, as expected.

Equivariant Chern Character

We now move on to equivariant K-theory and its corresponding Chern character. Let G be a compact Lie group acting on a space X. There is a naive equivariant Chern character which we may define in the following way

$$K_G(X, \mathbb{Q}) \to K_G(X \times EG, \mathbb{Q}) = K(X//G, \mathbb{Q}) = H^{even/odd}(X//G, \mathbb{Q}) = H_G^{even/odd}(X, \mathbb{Q})$$

(here $X//G = (X \times EG)/G$ is the homotopy quotient of X by G). However the first map is not in general an isomorphism. One reason why it should not be an isomorphism in general is that one expects the K-theory of a (connected) CW-complex Y to be complete: if we define $\mathcal{I} \subseteq K^0(Y)$ to be the kernel of the rank morphism $Y \to \mathbb{Z}$ then one expects

$$K^0(Y) = \varprojlim K^0(Y)/\mathcal{I}^N$$

by looking at the filtration by skeleta. This in particular applies to $Y = X//G$, so $K(X//G, \mathbb{Q})$ should be complete. But $K_G(X, \mathbb{Q})$ is not complete in general, for example when X is just a point.

It turns out that completeness is the only obstruction for the above map to be an isomorphism

Theorem (Atiyah-Segal completion theorem)**.** *$K^*(X//G)$ is the completion of $K_G(X)$ at the ideal $\mathcal{I} = \ker \operatorname{rank} \subseteq R_G$ of the representation ring of G.*

Example: Take $G = S^1$ and $X = pt$. Then $K^0_G(X) = R_G = \mathbb{Z}[L^{\pm}]$. Moreover,

$$K^0(X//G) = K^0(BS^1) = K^0(\mathbb{CP}^\infty) = \lim_{\leftarrow} K^0(\mathbb{CP}^n) = \mathbb{Z}[[L-1]]$$

as predicted by the completion theorem.

Observe that the complexified representation ring $R_G \otimes \mathbb{C}$ is the ring of class functions, that is, functions on $G_{\mathbb{C}}//G_{\mathbb{C}}$. Then $\mathcal{I}$ becomes the ideal defining the conjugacy class of 1. The natural question that we need to answer if we want to understand $K_G(X)$ globally is then: what are its completions at the other conjugacy classes? It turns out that there is a sequence of Chern characters, one for each conjugacy class, which together give a complete description of $K_G(X)$. We shall use the de Rham model to define them.

Equivariant Chern-Weil theory

From now on we let M be a manifold acted on by the group G and $E \to M$ be an (Hermitian) equivariant vector bundle. There is a de Rham model for equivariant cohomology called the Cartan model. This is a complex

$$((S^*\mathfrak{g}^* \otimes \Omega^* M)^G, d + \iota)$$

where $\mathfrak{g}^*$ has degree 2 and $\Omega^1 M$ has degree 1. This arises as the total complex a certain bicomplex

$$\begin{array}{ccc} (S^p\mathfrak{g}^* \otimes \Omega^q M)^G & \xrightarrow{d} & (S^p\mathfrak{g}^* \otimes \Omega^{q+1} M)^G \\ \downarrow \iota & & \downarrow \iota \\ (S^{p+1}\mathfrak{g}^* \otimes \Omega^{q-1} M)^G & \xrightarrow{d} & (S^{p+1}\mathfrak{g}^* \otimes \Omega^q M)^G \end{array}$$

Here the horizontal arrows are induced by the de Rham differential on M, and the vertical arrows are induced by the maps $\Omega^* M \to \mathfrak{g}^* \otimes \Omega^{*-1} M$ given by $\omega \mapsto \xi^a \iota_{\xi_a} \omega$, where ξ_a is a basis for $\mathfrak{g}$.

More generally, if we have a G invariant connection ∇ on E there is a curved complex $((S^*\mathfrak{g}^* \otimes \Omega^*(M,E))^G, \nabla^G)$, where $\nabla^G = \nabla + \iota$ is the total differential of the corresponding (curved) bicomplex. Then $(\nabla^G)^2$ defines an equivariant 2-form F^G with values in $\operatorname{End} E$, that is,

$$F^G \in (\mathfrak{g}^* \otimes \Gamma(\operatorname{End} E))^G \oplus (\Omega^2(M, \operatorname{End} E))^G$$

This is called the (equivariant) curvature of the connection. The naive Chern character may then be computed in this context as

$$ch(E) = \left[\operatorname{tr} \exp \frac{iF^G}{2\pi}\right]$$

Global equivariant Chern character

We are now ready to define the Chern character for other conjugacy classes

Theorem. *Let $[g]$ be a semisimple conjugacy class. Then the completion of $K^*_G(M,\mathbb{C})$ at $[g]$ is equal to $H^{even/odd}_{Z(g)}(M^g,\mathbb{C})$ where $Z(g)$ is the centralizer of g and M^g is the subset fixed by the minimally topologically cyclic subgroup of G whose complexification contains $[g]$. The isomorphism maps the class of an equivariant bundle E to*

$$\left[\operatorname{tr} g \exp \left.\frac{iF^G}{2\pi}\right|_{M^g}\right]$$

Observe that if $g = 1$ we recover the naive Chern character and the statement of Atiyah-Segal. In the case of G a finite group and $M = pt$, we get a chain of equalities

$$R_G \otimes \mathbb{C} = K^0_G(pt,\mathbb{C}) = \prod_{[g]} K^0_G(pt,\mathbb{C})^\wedge_{[g]} = \prod_{[g]} H^{even/odd}_{Z(g)}(pt,\mathbb{C}) = \prod_{[g]} \mathbb{C}$$

The resulting isomorphism $R_G \otimes \mathbb{C} = \prod_{[g]} \mathbb{C}$ is the usual isomorphism given by the character of a representation.

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