The Riemann-Hilbert Correspondence

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Let X be a smooth scheme over $\mathbb{C}$. We have two categories of interest: $\mathrm{DMod}(X)$, the category of D-Modules on X, and $\mathrm{Sh}(X)$ the category of sheaves of complex vector spaces on the analytic space underlying X. The goal of this talk is to establish an equivalence between appropriate subcategories of these. Morally, if we think of a D-module as a sheaf with a flat connection, we want to assign to it its sheaf of flat analytic sections.

One shouldn't expect this to give an equivalence without restricting what sheaves one considers: the category of D-modules knows information about the complex structure on X, while the category of sheaves only knows about the topology. Moreover, the category of sheaves is very non algebraic - it doesn't contain nontrivial compact objects, while $\mathrm{DMod}(X)$ is compactly generated.

The correspondence has a better chance of working when one restricts to D-modules which are close to flat vector bundles. It is a classical result that taking flat sections of a complex analytic vector bundle with a flat connection gives an equivalence

$$\mathrm{Conn}(X^{an}) = \mathrm{LocSys}(X)$$

with the category of local systems on X. This isn't true for algebraic flat vector bundles. For example, consider the following two connections on $\mathcal{O}_{\mathbb{A}^1}$:

$$\nabla_1(f) = \mathrm{d}\, f$$
$$\nabla_2(f) = \mathrm{d}\, f - f\,\mathrm{d}\, z$$

These are equivalent as analytic connections, via the change of gauge given by multiplication by e^z. They both have the same sheaf of flat sections, namely the constant sheaf. However they are not equivalent as algebraic connections, since ∇_1 has a flat algebraic section while ∇_2 doesn't. The problem here is that ∇_2 is a non-regular connection. In other words, the ordinary differential equation $f' - f = 0$ is not Fuchsian - infinity is an irregular singularity.

It is a theorem of Deligne that when one restricts to regular algebraic connections, one has the equivalence

$$\mathrm{Conn}_{\mathrm{reg}}(X) = \mathrm{LocSys}(X)$$

We want to go a bit further than this. We will work with holonomic D-modules, which are basically D-modules for which there is a stratification so that their restriction to each strata is a connection. On the topological side, we want to allow our local systems to differ between different strata - this is called a constructible sheaf. One expects then to have an equivalence

$$\mathrm{DMod}_{rh}(X) = \mathrm{Sh}_c(X)$$

between the categories of regular holonomic D-modules on X, and constructible sheaves on X. This is the Riemann-Hilbert correspondence, proven by Kashiwara in the context of complex manifolds and by Beilinson and Bernstein for smooth varieties.

Functoriality

We begin by discussing the different functors that we have on the side of D-modules. Everything will in principle be defined at the level of $\mathrm{DMod}(X)$, the (unbounded derived) category of quasicoherent D-modules. In some cases we will be able to restrict to its subcategory of compact objects, $\mathrm{DMod}_{\mathrm{coh}}(X)$.

Recall that we can think of D-modules as either left or right D-modules. The way we go from one description to the other is by tensoring with the dualizing sheaf ω_X. It is in general more convenient to think about them as right D-modules, and the underlying $\mathcal{O}_X$-module as being an ind-coherent sheaf on X. Our notation follows the conventions of [1] and [2].

Given a map $f : X \to Y$, we can define a pullback $f^! : \mathrm{DMod}(Y) \to \mathrm{DMod}(X)$. In terms of left D-modules, this is the ordinary pullback of the underlying quasicoherent sheaf, with its natural D_X action. In terms of right D-modules, this becomes the !-pullback at the level of the underlying ind-coherent sheaves, with its natural right D_X-module structure.

We can also define a pushforward $f_* : \mathrm{DMod}(X) \to \mathrm{DMod}(Y)$. As usual, functors between categories of modules can be presented by bimodules, and this isn't the exception. One can define the transfer bimodule $D_{X\to Y}$ to be the pullback of D_Y over f, as a quasicoherent sheaf. This is a $D_X - f^{-1}D_Y$-bimodule, and one can define the pushforward of a right D_X-module M to be the sheaf theoretic pushforward of $M \otimes_{D_X} D_{X\to Y}$, with its natural right D_Y-module structure.

We have a monoidal structure on $\mathrm{DMod}(X)$, given by tensoring the underlying $\mathcal{O}_X$-modules. We shall denote it by $\otimes^!$. There is a self-duality on the category $\mathrm{DMod}(X)$ given by

$$(M, N) = \Gamma_{dr}(M \otimes^! N))$$

where Γ_{dr} is the pushforward to the point. At the level of compact objects, this induces the Verdier duality functor

$$\mathbb{D} : \mathrm{DMod}_{\mathrm{coh}}(X) = \mathrm{DMod}_{\mathrm{coh}}(X)^{op}$$

satisfying $\mathrm{Hom}(\mathbb{D}M, N) = (M, N)$. Concretely, this is given by the formula

$$\mathbb{D}(M) = \mathscr{H}om_{D_X}(M \otimes_{\mathcal{O}_X} \omega_X^{-1}, D_X)$$

for M a right D_X-module.

The above defined operations are everything we have in this level of generality. They satisfy the usual rules, namely one has

- Adjunctions: when f is proper, f_* is left adjoint to $f^!$. When f is smooth, f_* is right adjoint to $f^![-2d]$, for d the relative dimension.
- Base change for f_* and $f^!$.

- Projection formula. In other words, f_* and $f^!$ are dual functors.
- The shriek pullback respects the monoidal structure.
- $\mathbb{D}^2 = \mathrm{id}$.

We skip the proof of all this. Most proofs are based on two tricks. The first trick is to split the map f into the composition of an immersion plus a projection from a product. The first case looks like the inclusion of a coordinate plane inside $\mathbb{A}^n$. The second one can be reduced to projection from projective space. Then one can understand these two cases separately. The second trick that one may rely on to prove some of these identities is to prove it on generators of our category, which when working on an affine scheme can be taken to be just the ring D itself. See [3] for details. Another route is to get everything as a particular case of the general machinery of ind-coherent sheaves on inf-schemes, as in [2].

Some computations

We now do some examples which show how to compute the functors mentioned in the previous section.

Consider first the inclusion $i : \mathbb{A}^1 \to \mathbb{A}^2$ of the affine line into the x-axis. Then the transfer module $D_{\mathbb{A}^1 \to \mathbb{A}^2}$ is equal to $D_{\mathbb{A}^1} \otimes \mathbb{C}[\partial_y]$ as a $D_{\mathbb{A}^1}$-module. In terms of left D-modules, this implies

$$i_* \mathcal{O}_{\mathbb{A}^1} = \varinjlim \mathcal{O}_{\mathbb{A}^2}/y^n[-1] = \mathbb{C}[\partial_y] \otimes \mathbb{C}[x][-1]$$

As expected, the effect of the pushforward is to formally extend the module to make ∂_y act (together with a shift which comes from the passage between left and right). Of course one should expect the result to be different from the structure sheaf of the x-axis, since points infinitesimally close to the x-axis are supposed to have the same fiber as points lying on the axis. What we are seeing is a sort of (shifted) structure sheaf of the formal completion of the x-axis.

Consider now the projection $\pi : X \to pt$ from X an n-dimensional smooth variety to the point. Then the transfer bimodule $D_{X \to pt}$ is simply $\mathcal{O}_X$, which has a resolution of the form

$$D_X \otimes_{\mathcal{O}_X} \Lambda^n T_X \to D_X \otimes_{\mathcal{O}_X} \Lambda^{n-1} T_X \to \ldots \to D_X$$

This can be thought of as a quantized version of the Koszul resolution of the structure sheaf of the zero section inside the cotangent bundle of X. Therefore one sees that the pushforward of a (right) D-module M is computed by the global sections of the complex

$$M \otimes_{\mathcal{O}_X} \Lambda^n T_X \to M \otimes_{\mathcal{O}_X} \Lambda^{n-1} T_X \to \ldots \to M$$

In terms of left D-modules, one obtains the de Rham complex of M:

$$M \to M \otimes_{\mathcal{O}_X} \Omega^1_X \to \ldots \to M \otimes_{\mathcal{O}_X} \Omega^n_X$$

shifted so that the term M is in degree $-2n$.

As a particular case, observe that

$$\pi_*\pi^!\mathbb{C} = \pi_*\mathcal{O}_X = H_{dr}(X)[2n]$$

The term $\pi_*\pi^!\mathbb{C}$ is what we could call the algebraic Borel-Moore homology of X, and the above computation is what we expect from Poincaré duality.

Finally, let's compute $\mathbb{D}\omega_X$. This is

$$\mathscr{H}om_{D_X}(\mathcal{O}_X, D_X)$$

and we can use the previously mentioned resolution of $\mathcal{O}_X$ to get

$$\mathbb{D}\omega_X = D_X \to D_X \otimes_{\mathcal{O}_X} \Omega^1_X \to \ldots \to D_X \otimes_{\mathcal{O}_X} \Omega^n_X$$

where the term D_X is in degree 0. This coincides with $\Omega^n_X[-n] = \omega_X[-2n]$. Therefore we see

$$\mathbb{D}\omega_X = \omega_X[-2n]$$

again agreeing with Poincaré duality (in fact this might as well be the statement of Poincaré duality).

More Functoriality

We now want to discuss the functors f^* and $f_!$. These are supposed to be left adjoint to f_* and $f^!$. Since f_* and $f^!$ are dual functors, it is a general fact that their left adjoints, if they exist, are given by

$$f_! = \mathbb{D}f_*\mathbb{D}$$

and

$$f^* = \mathbb{D}f^!\mathbb{D}$$

However, in order to be able to do this, one needs f_* and $f^!$ to preserve coherence, which doesn't happen in general: the shriek pullback of D_X to a point is an infinite dimensional vector space, and the shriek pushforward of D_X to a point computes cohomology of the structure sheaf of X, which is also infinite dimensional in general. One may see in fact that the sought after adjoints don't exist in those situations.

The situation is better if we restrict to the full subcategory $\mathrm{DMod}_h(X) \subset \mathrm{DMod}_{\mathrm{coh}}(X)$ of holonomic D-modules. It may be shown that f_*, $f^!$ and $\mathbb{D}$ preserve holonomicity. Therefore when working with holonomic D-modules one has the full six operations formalism.

We may now reinterpret the final computation from the last section as telling us

$$\pi^*\mathbb{C} = \pi^!\mathbb{C}[-2n]$$

and therefore

$$\pi_*\pi^*\mathbb{C} = \pi_*\pi^!\mathbb{C}[-2n]$$

which is in fact Poincaré duality, if we call the left hand side cohomology and the right hand side (shifted) Borel-Moore homology. Observe that the left hand side is nothing else than the algebraic de Rham cohomology of X. We shall see that the Riemann-Hilbert correspondence is compatible with the functoriality in both categories. This implies for example that algebraic de Rham cohomology is the same as the cohomology of the underlying manifold.

Constructible Sheaves

We now want to take a look at the second category involved in the Riemann-Hilbert correspondence. Let X be a smooth variety. To say what a constructible sheaf is, we need to agree on a notion of stratification of X. We shall use the following: a stratification $\mathcal{S}$ is a decomposition

$$X = \coprod X_\alpha$$

into subvarieties of X so that the closure of each stratum is a union of strata. A constructible sheaf for $\mathcal{S}$ is a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ (the unbounded derived category of sheaves on X, or better the category of sheaves of $\mathbb{Z}$-module spectra) such that its restriction to each stratum is a local system (i.e., a locally constant sheaf with finite stalks).

We denote by $\mathrm{Sh}_c(X)$ the category of sheaves constructible for some stratification. It may be shown that the usual six operations formalism on sheaves restricts well to the constructible category, so we have the same functoriality as with D-modules (cf. Chevalley's constructibility theorem).

The de Rham functor

We are now ready to introduce the functors relating D-modules to constructible sheaves. Let $M \in \mathrm{DMod}_h(X)$. Define

$$\mathrm{Sol}(M) = \mathscr{H}om_{D_X^{an}}(M^{an}, \mathcal{O}_X^{an})$$

where we take M to be a left module. It may be shown that this belongs to $\mathrm{Sh}_c(X)$: this is Kashiwara's constructibility theorem. The resulting functor

$$\mathrm{Sol} : \mathrm{DMod}_h(M)^{op} \to \mathrm{Sh}_c(M)$$

is called the functor of solutions. Its name comes from the fact that if M is associated to a system of partial differential equations on X, then $\mathrm{Sol}(M)$ is the sheaf of solutions of the system.

We will be interested in a second, covariant functor, given by

$$\mathrm{DR}(M) = M^{an} \otimes_{D_X^{an}} \mathcal{O}_X^{an}$$

where we take M to be a right module here. This is called the de Rham functor, and it is related to the functor of solutions by the identity $\mathrm{DR}(M) = \mathrm{Sol}(\mathbb{D}M)$. In particular, Kashiwara's constructibility theorem also applies for this functor, so we have

$$\mathrm{DR} : \mathrm{DMod}_h(X) \to \mathrm{Sh}_c(X)$$

The way we compute the de Rham functor is by using the previously mention resolution of $\mathcal{O}_X$. We have

$$\mathrm{DR}(M) = M^{an} \otimes_{\mathcal{O}_X^{an}} \Lambda^n T_X^{an} \to M^{an} \otimes_{\mathcal{O}_X^{an}} \Lambda^{n-1} T_X^{an} \to \ldots \to M^{an}$$

where the term M^{an} is in degree 0. In terms of left D-modules, this is the analytic de Rham complex

$$\mathrm{DR}(M) = M^{an} \to M^{an} \otimes_{\mathcal{O}_X^{an}} \Omega^1_{X^{an}} \to \ldots \to M^{an} \otimes_{\mathcal{O}_X^{an}} \Omega^n_{X^{an}}$$

shifted so that the term M^{an} is in degree $-2n$.

There are two main related problems that make the de Rham functor not be an equivalence:

- Injectivity fails already for connections for the reasons mentioned in the introduction. Namely,

$$\mathrm{DR}(D_{\mathbb{A}^1}/(\partial_z - 1)D_{\mathbb{A}^1}) = \mathrm{DR}(D_{\mathbb{A}^1}/\partial_z D_{\mathbb{A}^1})$$

  but those are two different $D_{\mathbb{A}^1}$-modules.

- DR doesn't commute with pullbacks or pushforwards.

The notion of regular holonomic D-module is meant to solve these two issues. We do however have some good behavior for the de Rham functor even before assuming regularity:

- DR commutes with Verdier duality.
- DR commutes with exterior tensor products.
- The singular support of $\mathrm{DR}(M)$ coincides with the characteristic variety of M.
- When f is non characteristic, one does have $\mathrm{DR}\, f^* = f^*\, \mathrm{DR}$. This is the content of the Cauchy-Kowalevsky-Kashiwara theorem. Morally, the content of this theorem is that when f is non characteristic with respect to a D-module, then the D-module looks like a connection in the directions normal to f, so locally to give a flat section one only needs to give a flat section along f.
- When f is proper, one does have $\mathrm{DR}\, f_* = f_*\, \mathrm{DR}$. This is proven by GAGA: we think of $\mathrm{DR}\, f_* M$ as having to do with algebraic flat sections of M along the fibers of f, and $f_*\, \mathrm{DR}\, M$ as having to do with analytic flat sections of M along the fibers. But if the fibers are proper, then both notions coincide.

This last point indicates that the notion of regular D-module is somehow allowing for a passage from the analytic world to the algebraic world. We had already seen a hint of this when we mentioned that Riemann-Hilbert in particular implies that algebraic and analytic de Rham cohomology agree.

Regularity

We call $M \in \mathrm{DMod}_h(X)$ a regular holonomic D-module if $i^! M$ is a regular connection for all $i : C \to M$ smooth curves for which $i^! M$ is a connection. There are more concrete criteria for when a D-module is regular, all having to do with the behavior of connections at infinity. We refer to [3] for details on this, here we shall content ourselves with stating the properties that we need to make Riemann Hilbert work:

Theorem. *The category* $\mathrm{DMod}_{rh}(X)$ *of regular holonomic* D*-modules on* X *is stable under all operations.*

More importantly, the following theorem contains the crucial passage from the analytic to the algebraic world. This solves the two problems mentioned in the previous section.

Theorem.

- *The functor* $\mathrm{DR} : \mathrm{DMod}_{rh}(X) \to \mathrm{Sh}_c(X)$ *commutes with all operations.*
- *(Deligne's Riemann Hilbert) The de Rham functor gives an equivalence*

$$\mathrm{Conn}_{\mathrm{reg}}(X) = \mathrm{LocSys}(X)$$

between the categories of regular connections on X *and local systems on* X*.*

The Riemann-Hilbert Correspondence

We are now ready to state and prove the correspondence (although most of the hard work actually goes into showing that the notion of regularity satisfies the properties mentioned in the previous section, so we are cheating a bit here).

Theorem. *The functor* $\mathrm{DR} : \mathrm{DMod}_{rh}(X) \to \mathrm{Sh}_c(X)$ *is an equivalence.*

Proof. We first show it is fully faithful. Let M, N be two regular holonomic D-modules. We have

$$\mathrm{Hom}(M, N) = \Gamma_{dr}(\mathbb{D}M \otimes^! N) = \Gamma_{dr}(\Delta^!(\mathbb{D}M \boxtimes N))$$

The same formalism is present on the constructible side, so we also have

$$\mathrm{Hom}(\mathrm{DR}(M), \mathrm{DR}(N)) = \Gamma(\Delta^!(\mathbb{D}\,\mathrm{DR}(M) \boxtimes \mathrm{DR}(N)))$$

The fact that DR commutes with all operations, then implies that

$$\mathrm{Hom}(M, N) = \mathrm{Hom}(\mathrm{DR}(M), \mathrm{DR}(N))$$

which means that DR is fully faithful.

We now show surjectivity. Let $Y \subset X$ be a smooth subvariety. Let $L \in \mathrm{LocSys}(Y)$ be a local system on Y. By Deligne's Riemann-Hilbert, we may find a connection M on Y such that

$$\mathrm{DR}_Y(M) = L$$

Then, denoting by i_Y the inclusion from Y to X, we have

$$\mathrm{DR}(i_{Y*}M) = i_{Y*}L$$

Our theorem then follows from the fact that the sheaves of the form i_*L generate the category $\mathrm{Sh}_c(X)$. To see this, let $\mathcal{F}$ be an arbitrary constructible sheaf, and let $j : Z \to X$ be the inclusion of the smooth locus of the support of $\mathcal{F}$. Then $j_*j^*\mathcal{F}$ is the star extension of a local system on a smooth subvariety, and the fiber of the map $\mathcal{F} \to j_*j^*\mathcal{F}$ has support strictly smaller than the support of $\mathcal{F}$. The result then follows by induction. □

Perverse Sheaves

So far our whole discussion happened at the level of derived categories. We now consider the t-structure on $\mathrm{DMod}_{rh}(X)$ coming from the forgetful functor from right D-modules to quasi-coherent sheaves (so that ω_X is in degree $-n$). We also have a t-structure on $\mathrm{Sh}_c(X)$ which makes sheaves whose stalks are classical vector spaces be in the heart. It turns out that the de Rham functor is not t-exact. For example, observe that

$$\mathrm{DR}(\omega_X) = \mathrm{DR}(\pi^!\mathbb{C}) = \pi^!\mathbb{C} = \mathbb{C}_X[2n]$$

More abstractly, one can show that Verdier duality preserves the t-structure on D-modules, but it fails to preserve the t-structure on sheaves (not even up to a shift). Therefore, we may transport the t-structure on D-modules to get an interesting t-structure on constructible sheaves, called the perverse t-structure. Its heart is an abelian category $\mathrm{Perv}(X)$, whose objects are called perverse sheaves. They admit the following concrete description

Theorem. *A sheaf $\mathcal{F}$ is perverse if and only if*

- $\dim \operatorname{supp} H^{-j}(\mathcal{F}) \leq j$ *for all* j
- $\dim \operatorname{supp} H^{-j}(\mathbb{D}\mathcal{F}) \leq j$ *for all* j

In terms of stalks, this means that generically the stalks are concentrated in degree $-n$, in codimension one they are allowed to concentrate in degrees $-n$ and $-n+1$, etc, until in codimension n they are allowed to concentrate in degrees $-n, -n+1, \ldots, 0$. The picture for costalks is similar only that they are concentrated in positive degrees instead of negative.

A simple argument considering characteristic cycles on the D-modules side shows that $\mathrm{DMod}_{rh}(X)^{\heartsuit} = \mathrm{Perv}(X)$ is an Artinian category. Its simple objects are in correspondence with pairs of a smooth subvariety Y together with an irreducible local system L on Y (where we identify two such pairs if they agree generically). The corresponding simple D-module is the unique simple extension of $L[\dim(Y)]$, and is denoted by

$$i_{!*}L[\dim(Y)]$$

This is called the middle extension of $L[\dim(Y)]$, and may be computed by a series of star extensions and truncations, see [3].

In our situation we are starting with a shifted local system on Y, so the resulting sheaf is also called an intersection cohomology sheaf, and denoted $IC_X(Y, L)$. It may be shown that

$$\mathbb{D}IC_X(Y, L) = IC_X(Y, L^\vee)$$

which is the statement of Poincaré duality for intersection cohomology sheaves. In the case when $Y = X$ and L is the constant sheaf, one recovers the usual Poincaré duality $\mathbb{D}(\mathbb{C}_X[n]) = \mathbb{C}_X[n]$.

References

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