The Geometric Satake Equivalence

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The classical Satake correspondence identifies the spherical Hecke algebra of a reductive group over a local ring with the representation ring of its Langlands dual group. This is the key piece which allows one to pass between the Galois and automorphic sides of the Langlands correspondence for function fields. In this talk we will discuss the geometric Satake correspondence, which plays a similar role in the geometric Langlands program

The Hecke category

Throughout the talk we let X be a smooth projective curve over $\mathbb{C}$ and G be a reductive algebraic group over $\mathbb{C}$ (for instance $GL_n, SL_n, PGL_n, SO_n, SP_n, ...$).

We are interested in the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles on X. As usual in algebraic geometry, we may study this space by studying functions that may be defined on it. Now, there are several things that may be called functions. The first thing that comes to mind is (sections of) the structure sheaf $\mathcal{O}_{\mathrm{Bun}_G(X)}$ which we may think of as complex valued functions. A bit less naively, one may consider the category $\mathrm{QCoh}_{\mathrm{Bun}_G(X)}$ of quasicoherent sheaves on $\mathrm{Bun}_G(X)$. If we think about a quasicoherent sheaf as a generalized vector bundle, where we allow fiber dimensions to jump, then we see that a quasicoherent sheaf may be thought of as a function which assigns a vector space to each point of $\mathrm{Bun}_G(X)$.

The usual operations with functions have analogues in the world of quasicoherent sheaves: one may add two quasicoherent sheaves and also multiply (tensor) them. In that way, $\mathrm{QCoh}_{\mathrm{Bun}_G(X)}$ has the structure of a symmetric monoidal category, which may be thought of as a categorification of the notion of a commutative algebra. Moreover, given an (appropriate) map between two varieties (or stacks), one may pullback and pushforward sheaves, in the same way that one may pullback functions and integrate functions along fibers. We may call QCoh a "function theory" since it behaves in a similar way as the usual theory of complex valued functions.

We are going to go a step further and instead of $\mathrm{QCoh}\,\mathrm{Bun}_G(X)$ we will consider $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ the category of D-modules on X. We may think of a D-module as a sheaf with a connection, and in that way it is just a vector space valued function together with extra structure. In particular, we may perform the operations discussed above, so D-modules are also a theory of functions.

Goal: Understand the category $\mathrm{DMod}\,\mathrm{Bun}_G(X)$.

The geometric Langlands equivalence is, as we shall see, an equivalence between this category and a certain other category.

We first need to make precise what it means to understand $\mathrm{DMod}\,\mathrm{Bun}_G(X)$. It turns out that there is a family of symmetries of this category, called point modifications, which depend on a choice of a point in X. Let $x \in X$ and let X' be the curve $X \cup \{x'\}$ consisting of X with a double point at x. In order to give a G-bundle on X' one must give a pair of bundles on X together with an identification away from x. That means that $\mathrm{Bun}_G(X') = \mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(X-x)} \mathrm{Bun}_G(X)$.

We may then give $\mathrm{DMod}\,\mathrm{Bun}_G(X')$ a monoidal structure (different from the usual commutative one) via the convolution product. Moreover, this category acts on $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ by the formula

$$H \cdot E = p_{2*}(H \otimes p_1^* E)$$

for $H \in \mathrm{DMod}\,\mathrm{Bun}_G(X')$ and $E \in \mathrm{DMod}\,\mathrm{Bun}_G X$. This is entirely analogous to the fact that complex valued functions on $\{1, \ldots, n\} \times \{1, \ldots, n\}$ form an algebra (the matrix algebra) which acts on complex valued functions on $\{1, \ldots, n\}$. Indeed, the only thing one needs to carry out that construction is a function theory.

Now, the category $\mathrm{DMod}\,\mathrm{Bun}_G(X')$ has a certain global nature; we would like to pass to a smaller category which only depends on what happens close to x. We are therefore lead to consider the formal disk $D_x = \mathrm{Spec}\,\mathbb{C}[[t]]$ around x, where t is a uniformizer, and the ravioli $D_x = D_x \cup \{x'\}$ which is the disk with a double center. In the same way as above, the category $\mathrm{DMod}\,\mathrm{Bun}_G(D'_x)$ has a monoidal structure. Moreover, there is a canonical (monoidal) map

$$\mathrm{DMod}\,\mathrm{Bun}_G(D'_x) \to \mathrm{DMod}\,\mathrm{Bun}_G(X')$$

and so $\mathrm{DMod}\,\mathrm{Bun}_G(D'_x)$ also acts on our category of interest $\mathrm{DMod}\,\mathrm{Bun}_G(X)$.

Definition. $\mathrm{DMod}\,\mathrm{Bun}_G(D'_x) = \mathcal{H}_x$ *is called the Hecke category.*

Our objective for most of this talk will be to understand this category, since it is the natural object acting on our main category of interest $\mathrm{DMod}\,\mathrm{Bun}_G(X)$.

Example: Let's take $G = GL_1$. Then Bun_G is the stack of line bundles. Let's work at the level of isomorphism classes. Then we have

$$\mathrm{Bun}_G(D'_x) = H^1(D'_x, \mathcal{O}^\times_{D'_x})$$

This may be computed using the Čech cover given by the two branches of the disk, yielding the complex

$$1 \to \mathbb{C}[[t]]^\times \times \mathbb{C}[[t]]^\times \to \mathbb{C}((t))^\times \to 1$$

and so we see that $H^1(D_x, \mathcal{O}^\times_{D'_x}) = \mathbb{Z}$. A D-module on this space is just given by the collection of its fibers, and therefore $\mathcal{H}_x$ is the category of $\mathbb{Z}$-graded vector spaces. The action of $\mathcal{H}_x$ on $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ is then determined by the action of δ_n, the skyscraper sheaf concentrated at $n \in \mathbb{Z}$. This may be seen to be induced from the map

$$\begin{aligned} \mathrm{Bun}_G(X) &\to \mathrm{Bun}_G(x) \\ L &\mapsto L(nx) \end{aligned}$$

Hecke algebras and double cosets

We now give a second point of view on the Hecke category, as a categorification of the concept of a Hecke algebra.

Let $H \leq \mathcal{G}$ be groups. We may wish to consider at first just finite groups so what follows is all well defined and there are no technical issues, although the idea should hold in different contexts with more or less technicalities.

The basic question Hecke algebras answer is the following: what object acts on H-invariants of all $\mathcal{G}$-representations? This question, although informally stated, has a precise answer, namely the algebra $\mathrm{End}(-^H)$ of endomorphisms of the functor $-^H : \mathcal{G}-\mathrm{Rep} \to \mathrm{Vec}$ of G-invariants. That functor is represented by the induced representation $\mathrm{Ind}_H^{\mathcal{G}}(1)$, where 1 is the trivial H-representation, which may be described as the space $Fun(\mathcal{G}/H)$ of functions on $\mathcal{G}/H$. Therefore, our algebra of interest is $\mathrm{End}_{\mathcal{G}}(Fun(\mathcal{G}/H))$. Now, linear endomorphisms of $Fun(\mathcal{G}/H)$ are given by functions on $\mathcal{G}/H \times \mathcal{G}/H$ with the convolution product, and therefore it follows that G-invariant endomorphisms are given by $Fun(\mathcal{G}/H \times_{\mathcal{G}} \mathcal{G}/H) = Fun(H\backslash\mathcal{G}/H)$. That leads us to the following

Definition. *The Hecke algebra of $\mathcal{G}, H$ is $Fun(H\backslash\mathcal{G}/H)$ with the convolution product.*

Observe for example that when $H = 1$ we are just saying that the group algebra of $\mathcal{G}$ is the object that acts on all linear representations of $\mathcal{G}$.

We now relate this concept to the Hecke category.

Claim. $\mathrm{Bun}_G(D'_x)$ *is a double coset space.*

To see that, observe that $\mathrm{Bun}_G(D'_x)$ consists of pairs of bundles P_1, P_2 on the disk, together with an isomorphism α on the punctured disk. If we choose a trivialization for P_1, P_2 then α becomes an element of the gauge group $G(K_x)$, where $K_x = \mathbb{C}((t))$. The changes in trivialization are given by the group $G(O_x)$, where $O_x = \mathbb{C}[[t]]$. Quotienting out by this, we see that

$$\mathrm{Bun}_G(D'_x) = G(O_x)\backslash G(K_x)/G(O_x)$$

which may be thought of as a noncommutative version of the Čech argument for $G = GL_1$. It follows that $\mathcal{H}_x = \mathrm{DMod}(G(O_x)\backslash G(K_x)/G(O_x))$ is just a categorified Hecke algebra.

We may also explain the action of $\mathcal{H}_x$ on $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ from this point of view. In the same way that above, any point in $\mathrm{Bun}_G(X)$ may be trivialized away from finitely many points of X. One may also trivialize it on a formal disk around every point of X. The clutching functions give an element of the restricted product $\prod'_{y\in X} G(K_y)$, and quotienting out by the changes in trivializations one gets

$$\mathrm{Bun}_G(X) = G(\mathbb{C}(X))\backslash {\prod_{y\in X}}' G(K_y)/G(O_y)$$

In particular, $\mathrm{DMod}\,\mathrm{Bun}_G(X) = \mathrm{DMod}(G(\mathbb{C}(X))\backslash \prod'_{y\neq X} G(K_y)/G(O_y) \times G(K_x))^{G(O_x)}$ so $\mathrm{DMod}\,\mathrm{Bun}\,G(X)$ is indeed $G(O_x)$-invariants of a $G(K_x)$-representation.

Incidentally, the above formula gives us a link between this geometric theory and the arithmetic case: in classical Langlands the left hand side does not make sense, but one has an analogous object to the right hand side, namely $G(F)\backslash G(\mathbb{A}_F)/G(O_F)$ where F is a number field and $\mathbb{A}_F$ is its ring of adeles. Instead of having D-modules one has L_2 functions on this space - the passage from one to the other is via Grothendiecks function-sheaf dictionary together with the Riemann Hilbert correspondence.

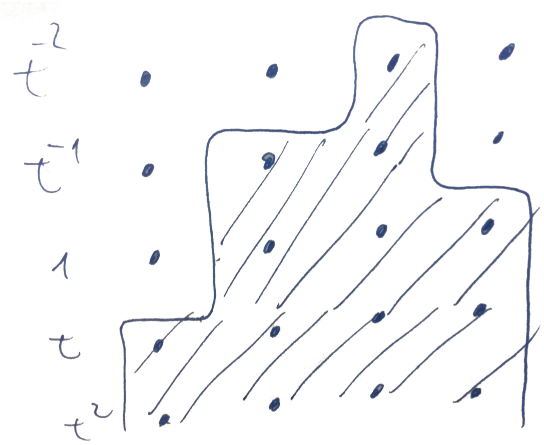
The Affine Grassmannian

Definition. *The affine Grassmannian is*

$$\mathrm{Gr}_{G,x} = G(K_x)/G(O_x)$$

This is supposed to be an analogue of the Grassmannian for loop groups. Observe that, based on the previous discussion, we have $\mathrm{Bun}_G(D'_x) = G(O_x)\backslash \mathrm{Gr}_{G,x}$. It turns out that the affine Grassmannian does not have stacky behavior, however it happens to be infinite dimensional. It may be shown in fact that it is an ind-scheme. We may also give a moduli interpretation for $\mathrm{Gr}_{G,x}$, as the moduli of bundles on the disk which are trivialized away from x.

Example: Let $G = GL_n$. We define a lattice to be an O_x-submodule L of K_x^n such that $t^M O_x^n \subset L \subset t^{-M} O_x^n$ for $M >> 0$. A typical picture of a lattice (where the dots represent the canonical basis of K_x^n as a $\mathbb{C}$-vector space) is



Observe that (the closed points of) $G(K_x)$ act transitively on the set of lattices, and moreover the stabilizer of O_x^n is $G(O_x)$. It follows that the closed points of $\mathrm{Gr}_{G,x}$ are in correspondence with lattices. This should give an idea as to why $\mathrm{Gr}_{G,x}$ is indeed an ind scheme, since for any fixed bound M the lattices form a (finite dimensional) projective variety.

Now, what are the $G(O_x)$-orbits in this example? It is not hard to see that every orbit has a unique decreasing lattice, corresponding to the class of a diagonal matrix with i-th diagonal entry t^{k_i}, where k_i are increasing. It follows that the set of orbits is given by $\mathbb{Z}^n/S_n$. Now something extraordinary happens: this set parametrizes irreducible representations of GL_n. Therefore we have a correspondence between (isomorphism classes of) points in $\mathrm{Bun}_G(D'_x)$ and irreducible representations of GL_n.

The geometric Satake equivalence generalizes that fact in two directions: it works for any reductive group, not only GL_n, and moreover it deals with the Hecke category instead of the orbits.

Let's first discuss what happens for other groups. Our first guess may be that $G(O_x)$-orbits should be in correspondence with irreducible representations of G, however that turns out to be wrong. Let $T \subset G$ be a maximal torus, $\Lambda_T^\vee$ and Λ_T be the lattices of cocharacters and characters, and let W be the Weyl group. Then what one has is that $G(O_x)$-orbits are in correspondence with Λ_T/W. However representations of G are classified by $\Lambda_T^\vee/W$, and Λ_T and $\Lambda_T^\vee$ may be different if G is not GL_n. The insight of Langlands was that Λ_T is the lattice of characters of a different group $G^\vee$, called the Langlands dual group of G. One therefore has a bijection

$$\{G(O_x)\text{−orbits}\} \leftrightarrow \{\text{Irreducible representations of } G^\vee\}$$

The group $G^\vee$ may be defined as the reductive group with dual root data to that of G. This duality fixes the families A, D and exceptionals, and switches B and C. At the level of topology, it switches the adjoint and simply connected forms. A few instances of this duality are

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
Sp_n	SO_{2n+1}
$Spin_{2n+1}$	$Sp_n/(\mathbb{Z}/2)$
$Spin_{2n}$	$SO_{2n}/(\mathbb{Z}/2)$

Satake

We are now ready to state the geometric Satake correspondence. It will be a categorification of the above statement about orbits. Let us first state the classical Satake isomorphism

Theorem (Satake 1963, Langlands)**.** *Let $q = p^n$ for some prime p. The spherical Hecke algebra*

$$\mathbb{C}[G(\mathbb{F}_q[[t]])\backslash G(\mathbb{F}_q((t))/G(\mathbb{F}_q[[t]])]$$

is isomorphic to the complexified representation ring of $G^\vee$.

This statement implies the statement about orbits that we stated before (although now in a function field context), but it also says something new about the product structure. Finally, we get to the geometric Satake correspondence:

Theorem (Lusztig, Drinfeld, Ginzburg, Mirkovic, Vilonen). *There is a monoidal equivalence*

$$\mathcal{H}_x = \operatorname{Rep} G^\vee$$

This is a Tannakian reconstruction theorem. One first shows that $\mathcal{H}_x$ has a fiber functor, is semisimple and symmetric monoidal. Then one has to work a bit more to show that the Tannakian group coincides with the Langlands dual group defined in terms of root data. One may also think about this theorem as giving a geometric definition of the group $G^\vee$.

The fiber functor for $\mathcal{H}_x$ is essentially the functor of solutions to D-modules. The symmetry is a bit more interesting: to prove it one has to construct a space which puts together all the affine Grassmannians as we vary the point x. This is a basic example of a factorization space. One then shows that the product in $\mathcal{H}_x$ is also obtained in a geometric way by colliding points in X. This fact allows one to show symmetry, in a similar way as the usual proof for the commutativity of the higher homotopy groups. The idea of factorization goes back to work of Beilinson and Drinfeld, however similar ideas had previously appeared in theoretical physics in the context of conformal field theory (which were later formalized in terms of vertex algebras). Indeed, linearizations of factorization spaces give factorization algebras (or chiral algebras) which encode the basic structure that one finds in quantum field theory.

Example: We go back to the case $G = GL_1$. Recall that $\mathcal{H}_x$ is the category of $\mathbb{Z}$-graded vector spaces. This is indeed equivalent to the category of GL_1-representations, since any representation splits into direct sum of one dimensional ones. This shows that $GL_1^\vee = GL_1$.

Geometric Langlands

We now go back to our initial object of interest, $\operatorname{DMod} \operatorname{Bun}_G(X)$. By the above theorem (in particular, the ideas of factorization!), we have a family of *commuting* operators $H_{V,x}$ called Hecke operators, indexed by a point $x \in X$ and a representation $V \in \operatorname{Rep} G^\vee$. The geometric Langlands correspondence may be thought of as a spectral decomposition of $\operatorname{DMod} \operatorname{Bun}_G(X)$ under this action. Equivalently, it gives a simultaneous diagonalization of the Hecke operators.

Let's allow ourselves to follow this idea rather informally to see where it leads. In this categorified context, we are supposed to find a space Z of eigenvectors, so that

$$\operatorname{DMod} \operatorname{Bun}_G(X) = \operatorname{QCoh}(Z)$$

The space Z should live above $\operatorname{Spec} \mathcal{H}$ where $\mathcal{H} = \bigotimes_{x \in X} \mathcal{H}_x$ is the total Hecke category, which we think of as being a categorified version of a commutative algebra. Now, let's assume that for each possible eigenvalue of the action there is exactly one eigenvector (usually called a Hecke eigensheaf). This means that $Z = \operatorname{Spec} \mathcal{H}$. Then the above equivalence becomes

$$\operatorname{DMod} \operatorname{Bun}_G(X) = \operatorname{QCoh}(\operatorname{Spec} \mathcal{H})$$

Now, the remaining question is, what is $\operatorname{Spec}\mathcal{H}$?. A point in that space is supposed to be some sort of map

$$\mathcal{H} \to \mathrm{Vec}$$
$$x \in X, V \in \operatorname{Rep} G^\vee \mapsto \text{vector space}$$

Equivalently, that is a map

$$V \in \operatorname{Rep} G^\vee \mapsto (x \in X \mapsto \text{vector space})$$

The object on the right is just a vector bundle on X. It turns out in fact that one has to consider flat vector bundles on the right (the reason for this having to do with what we mean exactly by $\bigotimes_{x\in X} \mathcal{H}_x$). Therefore points in $\operatorname{Spec}\mathcal{H}$ are machines that take in representations of $G^\vee$ and give out flat vector bundles. These are just $G^\vee$-local systems, so we have $\operatorname{Spec}\mathcal{H} = \mathrm{LocSys}_{G^\vee}$. Putting everything together, we have the following statement:

Conjecture (Naive Geometric Langlands Correspondence)**.** *There is an equivalence*

$$\mathrm{DMod}\,\mathrm{Bun}_G\, X = \mathrm{QCoh}\,\mathrm{LocSys}_{G^\vee}\, X$$

which commutes with the action of the Hecke operators.

The left side of the correspondence is called the automorphic or geometric side, and the right side is the Galois or spectral side. This conjecture as it is is known to be wrong. To get a statement that has a chance to be true, one has to take everything to be derived, and replace coherent sheaves on the right by ind coherent sheaves with a certain singular support.

References

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