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Métodos Cohomológicos en Ecuaciones Diferenciales

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Introducción

El objetivo de esta tesis es presentar algunas aplicaciones del álgebra homológica a las ecuaciones en derivadas parciales no lineales. Nos concentraremos en particular en la teoría formal de ecuaciones diferenciales, en la que se trabaja al nivel de series formales de potencias, sin requerir condiciones de convergencia. Varias construcciones homológicas aparecen naturalmente al estudiar sistemas de ecuaciones sobredeterminados, los cuales pueden poseer obstrucciones a la existencia de soluciones.

La teoría será desarrollada de manera intrínseca, usando espacios de jets. Este es el contexto natural para estudiar ecuaciones provenientes de la geometría, en las cuales no hay un sistema de coordenadas preferencial. Mas aún, este punto de vista es esencial para tratar problemas globales, en los cuales no existe la posibilidad de trabajar en coordenadas.

Sea M una variedad diferencial y fijemos $n \leq \dim M$. Diremos que dos subvariedades de dimensión n que contienen a un punto $q \in M$ definen el mismo jet de orden k en q si tienen orden de contacto al menos k en q. El espacio de jets de orden k de subvariedades de dimensión n en q se denota $J^k_n(M)_q$, y denotaremos $J^k_n(M) = \bigcup_{q \in M} J^k_n(M)_q$.

Una ecuación diferencial de orden k en las subvariedades de dimensión n de M es un subconjunto $R \subseteq J^k_n(M)$. Se puede pensar a R como una restricción en los posibles k-jets de subvariedades. Una solución de R es una subvariedad $N \subseteq M$ de dimensión n tal que su k-jet en todo punto pertenece a R.

Esto generaliza la definición usual en coordenadas, como sigue. Supongamos dado un sistema de ecuaciones diferenciales parciales para funciones $u : \mathbb{R}^n \to \mathbb{R}^s$

$$G\left(x, u(x), \frac{\partial u}{\partial x^I}(x)\right) = 0 \tag{1}$$

donde $G : \mathbb{R}^{n+s\binom{n+k}{n}} \to \mathbb{R}^\lambda$ e I recorre los multi-índices de longitud a lo sumo k. Puede verse que el sistema (1) define una ecuación diferencial $R \subseteq J^k_n(\mathbb{R}^n \times \mathbb{R}^s)$, cuyas soluciones son exactamente los gráficos de soluciones de (1).

Recíprocamente, si $x^1, \ldots, x^n, u^1, \ldots, u^s$ son coordenadas locales en una variedad M, podemos describir (casi todas) las subvariedades de M como el gráfico de una función $u : \mathbb{R}^n \to \mathbb{R}^s$. Cualquier ecuación diferencial $R \subseteq J^k_n(M)$ toma la forma (1) en este sistema de coordenadas.

Es común también desarrollar esta teoría empezando con una variedad fibrada $\xi : E_\xi \to M$ (o sea, una submersión suryectiva). En este caso, los objetos de interés son los espacios $J^k(\xi)$ de k-jets de secciones de ξ, y uno puede definir una ecuación diferencial en secciones de ξ como un subconjunto $R \subseteq J^k(\xi)$. Como antes, las secciones de ξ se identifican con ciertas subvariedades del espacio total E_ξ, así que nuestro punto de vista es mas general. Sin embargo, esto sigue siendo un caso muy importante. Por ejemplo, si empezamos con un fibrado trivial $\xi : M_1 \times M_2 \to M_1$, podemos hablar de ecuaciones diferenciales en funciones $f : M_1 \to M_2$. En el caso en que ξ es un fibrado vectorial, se puede desarrollar la teoría de ecuaciones diferenciales lineales en forma intrínseca.

Otro enfoque geométrico usado para estudiar ecuaciones en derivadas parciales está dado por los sistemas diferenciales exteriores. Estos son ecuaciones de primer orden muy particulares, definidas a partir de un ideal diferencial del álgebra de formas diferenciales en la variedad. Resulta que dada cualquier ecuación $R \subseteq J^k_n(M)$ de orden k, existe un sistema diferencial exterior definido sobre R cuyas soluciones están en correspondencia con las soluciones de R. Esta teoría fue usada con muy buenos resultados por Élie Cartan (quien, en particular, introdujo las formas exteriores de grado mayor que 3 y la derivada exterior), continuando trabajos previos de Pfaff, Frobenius y Darboux sobre el problema de Pfaff. Un desarrollo moderno de esta teoría puede encontrarse en [1].

La principal herramienta algebraica usada en esta tesis es la cohomología de Spencer de comódulos graduados sobre una coálgebra de polinomios. Dado un espacio vectorial V de dimensión n sobre un cuerpo de característica cero, y un comódulo graduado $\mathcal{A}$ sobre la coálgebra SV^*, pueden construirse ciertos espacios vectoriales graduados $H^q(\mathcal{A}) = \bigoplus_{k\geq q} H^{q,k}(\mathcal{A})$, la cohomología de Spencer de $\mathcal{A}$. Asumiendo condiciones de finitud, estos espacios son duales a la homología de Koszul del SV-módulo dual. Estos grupos fueron usados explícitamente por primera vez por Spencer[11, 12] en el contexto de deformaciones de estructuras dadas por pseudogrupos.

Dada una ecuación diferencial $R \subseteq J^k_n(M)$ de orden k puede definirse, para cada $l \geq 0$, el l-ésimo prolongado $R^{(l)}$, que es una ecuación diferencial de orden $k + l$. En coordinadas, esto se corresponde con agregarle al sistema las derivadas de orden a lo sumo l de sus ecuaciones. Decimos que R es formalmente integrable si la proyección $R^{(l)} \to R^{(l-1)}$ es una submersión suryectiva para todo $l > 0$. Cuando esto ocurre, uno puede construir soluciones formales a R (en coordenadas, series formales en un punto que satisfacen la ecuación y todas sus derivadas) empezando de cualquier punto de R (es decir, a partir de una solución infinitesimal).

El primer problema en la teoría formal de sistemas de ecuaciones sobredeterminados es el de construir las obstrucciones a que una ecuación sea formalmente integrable. Luego de trabajos de Bott y Quillen, Goldschmidt [5, 4] construyó las obstrucciones a la integrabilidad formal, que se encuentran en el segundo grupo de cohomología de Spencer de un cierto fibrado de comódulos asociado al símbolo de la ecuación.

En la categoría analítica, la integrabilidad formal implica la existencia de soluciones formales. Esto se sigue del δ-Poincaré estimate de Spencer, demostrado por Ehrenpreis, Guillemin y Sternberg [2], y mas adelante por Sweeney[13] usando técnicas distintas. En la categoría C^∞, la integrabilidad formal no garantiza la existencia de soluciones. Un ejemplo de esto fue dado por Lewy[8].

Otra pregunta fundamental es la existencia de soluciones al problema de valores iniciales. Supongamos dada una subvariedad $N_{n-1} \subseteq M$ de dimensión $n-1$, y condiciones iniciales a lo largo de N_{n-1} (o sea, una sección $N_{n-1} \to R$). Una solución al problema de valores iniciales es una solución N de R con $N_{n-1} \subseteq N$, que verifique las condiciones iniciales. Para que puedan existir soluciones, las condiciones iniciales deben satisfacer una cierta ecuación de primer orden (que corresponde, en coordenadas, con la conmutatividad de las derivadas parciales). Veremos que si dicha ecuación es formalmente integrable entonces las únicas obstrucciones para resolver el problema de valores iniciales formal con condiciones iniciales genéricas ocurren en el primer orden (es decir, al resolver el problema con condiciones iniciales dadas a lo largo de un 1-jet de una subvariedad de dimensión $n-1$).

En el caso analítico, la existencia de soluciones al problema de valores iniciales está garantizada por el teorema de Cartan-Kahler en la teoría de sistemas diferenciales exteriores, que depende del teorema de Cauchy-Kowalevski sobre existencia y unicidad para sistemas determinados de ecuaciones en derivadas parciales analíticas.

Un concepto central en la teoría es el de involución. Las ecuaciones que están en involución pueden ser resueltas, en el caso analítico, por una sucesión de problemas de valores iniciales, y el tamaño del espacio de soluciones puede ser estimado. Serre fue el primero en observar que la involutividad es también una condición homológica: es equivalente a la anulación de la cohomología de Spencer del comódulo asociado al símbolo de la ecuación en (casi) todo grado, junto con integrabilidad a primer orden (ver Guillemin-Sternberg[6]). De hecho, el concepto de involución resulta ser dual a la regularidad de Castelnuovo-Mumford en álgebra conmutativa, ver Malgrange[9].

Bajo condiciones bastante técnicas, cualquier ecuación diferencial se vuelve involutiva luego de suficientes prolongaciones, resultado que depende del teorema de la base de Hilbert. Esto es llamado el teorema de prolongación de Cartan-Kuranishi, conjeturado por Cartan y probado por Kuranishi[7] en el contexto de sistemas diferenciales exteriores. En esta tesis probaremos una versión débil de ese teorema: cualquier ecuación diferencial formalmente integrable puede prolongarse hasta ser involutiva. En particular, la existencia de soluciones a ecuaciones analíticas formalmente integrables se reducirá también al teorema de Cartan-Kahler.

Una importante aplicación de esta teoría es al problema de equivalencia de estructuras geométricas. Dada una estructura geométrica en una variedad (una métrica Riemanniana, estructura casi compleja, distribución, etc) se quiere encontrar un sistema completo de invariantes que (aparte de dar información útil sobre la geometría) permitan decidir si dos de esas estructuras son (localmente) equivalentes. El problema de decidir si dos variedades Riemannianas son localmente isométricas es un ejemplo básico de ésto.

Cartan desarrolló un método para calcular invariantes, usando la teoría de sistemas diferenciales exteriores. La observación básica es que las equivalencias entre dos estructuras resuelven una ecuación diferencial de primer orden. El método consiste en modificar las estructuras hasta que (si todos los invariantes coinciden) dicha ecuación se vuelve involutiva, lo que garantiza la existencia de equivalencias formales. Este proceso puede involucrar repetidas prolongaciones, y el hecho de que termina es consecuencia del teorema de la base de Hilbert. Entre las exposiciones modernas del método de Cartan podemos encontrar Gardner[3] y Olver[10].

Esta tesis está organizada de la siguiente manera:

El capítulo I contiene el formalismo básico de espacios de jets, desde el punto de vista de jets de subvariedades. Luego los resultados son adaptados al caso de jets de secciones de variedades fibradas. Consideraremos jets de secciones sobre subvariedades de codimensión positiva de la variedad base; nuestra principal motivación para esto es que estos espacios aparecen naturalmente al discutir linearización de ecuaciones y operadores no lineales.

El capítulo II discute la homología de Koszul y la cohomología de Spencer. También contiene una introducción al estudio algebraico de la involución.

El capítulo III trata sobre operadores diferenciales y ecuaciones. Nuevamente, presentamos todos los temas para espacios de jets de subvariedades, y después los adaptamos para jets de secciones. Construiremos las obstrucciones a la integrabilidad formal, demostrando así el teorema de Goldschmidt. Finalmente, discutiremos el problema de valores iniciales formal.

El capítulo IV discute el método de Cartan para resolver el problema de equivalencia. Haremos uso del formalismo de jets semi-holonómicos (mas específicamente, fibrados de marcos semi-holonómicos) para tratar el proceso de prolongación que requiere el método. El principal resultado en este capítulo es un teorema de equivalencia formal para G-estructuras de orden k. Probaremos también que el método termina.

Introduction

The aim of this thesis is to present several applications of homological algebra to nonlinear partial differential equations. We shall focus on the formal theory of differential equations, in which one works at the level of formal power series without imposing convegence conditions. Homological constructions arise naturally when studying overdetermined systems, which may contain obstructions to the existence of solutions.

The theory will be developed in a coordinate free way, using jet spaces. This is the natural framework for discussing PDE arising from questions in geometry, where there is no preferred coordinate system. Moreover, this approach is necessary for dealing with global problems where one does not have the option of working in coordinates.

Let M be a differentiable manifold and fix $n \leq \dim M$. Two n-dimensional submanifolds passing through a point $q \in M$ are said to have the same k-th order jet at q if they have order of contact at least k at q. The space of k-th order jets of n-dimensional submanifolds at q is denoted by $J^k_n(M)_q$, and one defines $J^k_n(M) = \bigcup_{q \in M} J^k_n(M)_q$.

A k-th order differential equation on n-dimensional submanifolds of M is a subset $R \subseteq J^k_n(M)$. This may be thought of as a restriction on the possible k-jets of submanifolds. An n-dimensional submanifold $N \subseteq M$ is called a solution of the equation if its k-th jet at each point belongs to R (that is, if it satisfies all the restrictions).

This generalizes the traditional, coordinate dependent definition of a partial differential equation, as follows. Suppose that we are given a system of k-th order PDE for maps $u : \mathbb{R}^n \to \mathbb{R}^s$

$$G\left(x, u(x), \frac{\partial u}{\partial x^I}(x)\right) = 0 \tag{2}$$

where $G : \mathbb{R}^{n+s\binom{n+k}{n}} \to \mathbb{R}^\lambda$ and I ranges over all multi-indices of length at most k. Equation (2) is seen to define a differential equation $R \subseteq J^k_n(\mathbb{R}^n \times \mathbb{R}^s)$, whose solutions are exactly the graphs of solutions of (2).

Reciprocally, when working in local coordinates $x^1, \ldots, x^n, u^1, \ldots, u^s$ on a manifold M, we may describe (most) n-dimensional submanifolds $N \subseteq M$ using graphs of functions $u : \mathbb{R}^n \to \mathbb{R}^s$. Differential equations $R \subseteq J^k_n(M)$ take the form (2) in this coordinate system.

It is also usual to develop the theory starting with a fibered manifold $\xi : E_\xi \to M$ (that is, a surjective submersion). In this case, the objects of interest are the spaces

$J^k(\xi)$ of k-jets of sections of ξ, and one may define differential equations acting on sections as subsets $R \subseteq J^k(\xi)$. As above, sections of ξ may be identified with certain submanifolds of E_ξ, and so our approach generalizes this. However, this remains an important particular case. For example, if one takes ξ to be a trivial bundle $M_1 \times M_2 \to M_1$, then one may speak of differential equations on functions $f : M_1 \to M_2$. One may also take ξ to be a vector bundle, in which case one may develop the theory of linear PDE in a coordinate free way.

Another common geometric approach to partial differential equations is via exterior differential systems. These are special first order equations defined using ideals of the algebra of differential forms, closed under exterior differentiation. It turns out that for any k-th order differential equation $R \subseteq J^k_n(M)$ there is an exterior differential system on R whose solutions are in correspondence with the solutions to R. This theory was used to great effect by Élie Cartan (who, in particular, introduced exterior forms of degrees greater than 3, and the exterior derivative), after previous work by Pfaff, Frobenius and Darboux on the Pfaff problem. See [1] for a modern treatment of this theory.

The main algebraic tool used in this thesis is the Spencer cohomology of graded comodules over a polynomial coalgebra. Given an n-dimensional vector space V over a field of characteristic zero and a graded comodule $\mathcal{A}$ over the coalgebra SV^*, one may construct certain graded vector spaces $H^q(\mathcal{A}) = \bigoplus_{k \geq q} H^{q,k}(\mathcal{A})$ called the Spencer cohomology of $\mathcal{A}$. Under finiteness conditions, these spaces are dual to the Koszul homology of the dual SV module. These groups were first explicitly used by Spencer [11, 12] in the context of deformations of pseudogroup structures.

Given a smooth k-th order differential equation $R \subseteq J^k_n(M)$, one may form the l-th prolongation $R^{(l)}$ which is a differential equation of order $k + l$. In coordinates, this corresponds to adjoining the derivatives of order at most l of the equation in question. A differential equation is said to be formally integrable if $R^{(l)} \to R^{(l-1)}$ is a smooth submersion for all $l > 0$. When this happens, one may construct formal solutions to the equation (in coordinates, formal power series satisfying the equation and all its derivatives) starting with any point in R.

The first problem in the formal theory of overdetermined systems is to construct the obstructions for an equation to be formally integrable. Following work by Bott and Quillen, Goldschmidt [5, 4] constructed the obstructions to integrability, which lie on the second Spencer cohomology of a certain bundle of comodules associated to the symbol of the differential equation.

In the analytic category, formal integrability implies the existence of local solutions. This follows from the δ-Poincaré estimate of Spencer, proved by Ehrenpreis, Guillemin and Sternberg [2], and later by Sweeney[13] by different means. In the smooth category, formal integrability does not guarantee the existence of solutions, as shown by Lewy[8].

Another fundamental question is the initial value problem. Suppose that we are given a $(n-1)$-dimensional submanifold $N_{n-1} \subseteq M$ and initial conditions along N_{n-1}

(i.e., a section $N_{n-1} \to R$). A solution to the initial value problem is then a solution N of R containing N_{n-1} and satisfying the initial conditions. In order for solutions to exist it is necessary that the initial conditions solve a certain first order equation (corresponding in coordinates to the commutativity of partial derivatives). We shall see that if the equation satisfied by the initial conditions is formally integrable then (under mild regularity hypothesis) the only obstructions to the formal initial value problem with generic initial conditions arise in the first order (that is, when solving the initial value problem with conditions given along a 1-jet of an $(n-1)$-dimensional submanifold).

In the analytic case, the existence of a solution to the initial value problem is guaranteed by the Cartan-Kahler theorem in the theory of exterior differential systems, which ultimately depends on the Cauchy-Kowalevski existence and uniqueness result for analytic partial differential equations.

A central concept in the theory is that of involution. Equations which are in involution may be solved (at least in the analytic case) by a sequence of initial value problems, and one may estimate the size of the solution space. It was observed by Serre that the involutivity of a differential equation is also of homological nature: it is equivalent to the vanishing, in almost all degrees, of the Spencer cohomology of the symbol comodule of the equation, together with integrability to first order (see Guillemin-Sternberg[6]). In fact, the concept of involution turns out to be dual to the Castelnuovo-Mumford regularity in commutative algebra, see Malgrange[9].

Under fairly technical conditions, any differential equation becomes involutive after enough prolongations, a result which ultimately depends on the Hilbert basis theorem. This is called the Cartan-Kuranishi prolongation theorem, conjectured by Cartan and proved by Kuranishi[7] in the context of exterior differential systems. In this thesis we shall prove a weak version of this theorem: any formally integrable differential equation may be completed to involution (i.e., prolonged until it becomes involutive). In particular, the existence of solutions to an analytic formally integrable equation may also be reduced to the Cartan-Kahler theorem.

One remarkable application of this theory is to the equivalence problem for geometric structures. Given a geometric structure on a manifold (a Riemannian metric, almost complex structure, distribution, etc.), one wants to compute a complete set of invariants of the structure, which (apart from giving useful information about the geometry) allows us to decide whether two such structures are (locally) equivalent. The problem of deciding when two Riemannian manifolds are isometric is a basic example of this.

Cartan developed a method for computing these invariants, using his theory of exterior differential systems. The basic observation is that the equivalences between two structures solve a certain first order differential equation. One then modifies the structures until (if all the invariants obtained coincide) the equation becomes involutive, thus establishing the existence of formal solutions. This process may involve repeated prolongation, and the fact that it terminates is, once more, a consequence of the Hilbert basis theorem. Modern expositions of Cartan's method include Gardner[3] and Olver[10].

This thesis is organized as follows:

Chapter I contains the basic formalism of jet spaces, from the point of view of jets of submanifolds. The results are then adapted to the case of jets of sections of a fibered manifold. We consider jets of sections over submanifolds of the base manifold of positive codimension; our main motivation for this is that they arise naturally when discussing linearization of nonlinear equations and operators.

Chapter II presents the necessary background on Koszul homology and Spencer cohomology. We also give an introduction to the algebraic theory of involution.

Chapter III discusses differential operators and equations. Again, everything is first presented for spaces of jets of submanifolds, and then adapted to jets of sections. The obstructions to formal integrability are constructed, proving Goldschmidt's theorem. We then discuss the formal initial value problem.

Chapter IV deals with Cartan's method for solving the equivalence problem. We make use of semi-holonomic jets (specifically, semi-holonomic frame bundles) in order to deal with the prolongation step in the method. The main result in this chapter is a formal equivalence theorem for semi-holonomic higher order G-structures. The fact that the method terminates is also proven.

Chapter I

Jet Spaces

This chapter deals with the basic theory of jet spaces, which is the framework that will allow us to discuss differential equations in a coordinate-free way.

In section 1 we define the space $J^k_n(M)$ of k-th order jets of n-dimensional submanifolds of a manifold M. These spaces come equipped with a universal bundle which generalizes the universal bundle on a Grassmannian. The main result in this section is the fact that the projections $J^k_n(M) \to J^{k-1}_n(M)$ are affine bundles for $k \geq 2$, modeled on the bundle of k-th order homogeneous polynomials from the universal bundle to the universal quotient bundle. The jet spaces behave (almost) functorially: any differentiable map between two manifolds induces a (partially defined) map between the jet spaces, called its prolongation. We shall see that prolongued morphisms respect the affine structure of the jet spaces, and give a precise description for the associated vector bundle maps. This will be generalized in chapter III when we study the prolongation of differential operators.

Section 2 deals with the spaces $J^k_n(\xi)$ of jets of sections of a fibered manifold $\xi : E_\xi \to M$ over n-dimensional submanifolds of M. The image of each section is an n-dimensional submanifold of the total space E_ξ, so this may be studied with the formalism of jets of submanifolds. In the case when $n = \dim M$, these are the spaces of jets of sections $M \to E_\xi$. Our main motivation for studying the spaces $J^k_n(\xi)$ for $n < \dim M$ is that they are needed in order to describe the tangent space to to the jet spaces $J^k_n(M)$. This description will allow us to define the linearization of nonlinear differential operators and equations in chapter III.

In section 3 we define the contact distribution $\mathcal{C}^k_n$ on the space $J^k_n(M)$. This distribution has the property that its n-dimensional integral submanifolds which are trasverse to the vertical distribution are in correspondence with the n-dimensional submanifolds of M. Unless $n = \dim M$, this distribution will not be Frobenius integrable. We shall study how the Frobenius condition fails: this is measured in terms of a certain (vector bundle valued) 2-form $\overline{[\cdot,\cdot]}$ on $\mathcal{C}^k_n$. The space $J^{k+1}_n(M)$ is recovered as the space of n-dimensional planes tangent to $\mathcal{C}^k_n$ such that the rescriction of $\overline{[\cdot,\cdot]}$ vanishes (together with a trasversality condition). This fundamental fact implies that the spaces $J^k_n(M)$ may be studied in a purely inductive manner.

1 Jets of Submanifolds

1.1 Let M be a differentiable manifold, and fix $n \leq \dim M$. Let N be an (inmersed) n-dimensional submanifold of M, and let $q \in N$. Then N defines an ideal $I_q(N) \subset \mathcal{O}_{M,q}$ of the ring of germs of smooth functions on M at q, given by those germs that vanish when restricted to N. This ideal completely characterizes the germ of N at q: if we are given another submanifold such that the corresponding ideals agree, then the submanifolds agree near q. We shall adapt this idea to define what it means for two submanifolds to have order of contact at least k at q.

Let $k \geq 0$. Denote by $\mathrm{m}_{M,q}$ the maximal ideal of $\mathcal{O}_{M,q}$. Then $\mathcal{O}_{M,q}/\mathrm{m}^{k+1}_{M,q}$ is called the ring of *k-jets* of functions at q with values in $\mathbb{R}$. That is, we are identifying two functions defined around q if they differ by terms of order at least k. In coordinates, two functions are identified if their derivatives agree up through order k, and we are left with the possible k-th order Taylor expansions.

The inclusion of a submanifold N induces a restriction map $i^* : \mathcal{O}_{M,q}/\mathrm{m}^{k+1}_{M,q} \to \mathcal{O}_{N,q}/\mathrm{m}^{k+1}_{N,q}$. Let $I^k_q(N)$ be the kernel of i^*. Then $I^k_q(N)$ is the ideal of k-jets of functions at q that vanish when restricted to N. Given another n-dimensional submanifold N' which passes through q, we say that N and N' have the same *k-jet* at q if $I^k_q(N) = I^k_q(N')$. This defines an equivalence relation on the set of submanifolds passing through q. We denote by $J^k_n(M)_q$ the set of k-jets of n-dimensional submanifolds passing through q, and $J^k_n(M) = \bigcup_{q\in M} J^k_n(M)_y$ the set of k-jets of n-dimensional submanifolds of M. For $j \leq k$ we have projections $\pi_{k,j} : J^k_n(M) \to J^j_n(M)$. If $y \in J^k_n(M)$, we usually denote by y_j its projection to $J^j_n(M)$.

Notice that $J^0_n(M) = M$. Also, $I^1_q(N) \subseteq \mathrm{m}_{M,q}/\mathrm{m}^2_{M,q} = T^*_qM$ is the annihilator of T_qN. Thus $J^1_n(M)_q$ is the Grassmannian of n-dimensional subspaces of T_qM, and $J^1_n(M)$ is the Grassmannian bundle associated to TM.

1.2 Let $x^1, \dots x^n, u^1, \dots, u^s$ local coordinates for M around q, and let p be the projection of q to the space of coordinates x^i. Suppose that N is given by the graph of a function f in those coordinates. Then N is the zero locus of the functions $u^i - f^i \circ \xi$, where f^i are the coordinates of f, and ξ is the projection to the first n coordinates. The ideal $I^k_q(N)$ is then generated by those s functions. As discussed before, the ring $\mathcal{O}_{M,q}/\mathrm{m}^k_{M,q}$ is isomorphic to $\mathbb{R}[x^1, \dots x^n, u^1 \dots u^s]/\langle x^1, \dots x^n, u^1, \dots, u^s\rangle^k$, sending each function to its Taylor expansion at q. It follows that the ideal $I^k_q(N)$ only depends on the k-th order Taylor expansion of f at p. If N' is another submanifold passing through q and given locally by the graph of f', then N and N' have the same k-jet at q if and only if the derivatives of f and f' agree up through order k at p.

We are now ready to define the differentiable structure on $J^k_n(M)$ for $k \geq 1$. Pick $x^1, \ldots, x^n, u^1, \ldots, u^s$ local coordinates on M. The jet of a submanifold transverse to the vertical depends, as discussed above, on the k-th order Taylor expansion of a function around a point $(x^1, \ldots, x^n)$. Thus we can introduce coordinates x^i, u^a_I on $J^k_n(M)$, where $1 \leq i \leq n$, $1 \leq a \leq s$, and I is a symmetric multi-index with $0 \leq |I| \leq k$. Explicitly, the jet of the graph of a function f around a point (x^i, u^a) has coordinates x^i and $u^a_I = \frac{\partial^{|I|}}{\partial x^I} f(x^i)$. This defines a differentiable structure on $J^k_n(M)$. The projection $\pi_{k+1,k} : J^{k+1}_n(M) \to J^k_n(M)$ is given in those coordinate systems by forgetting some coordinates, so it is a surjective submersion.

1.3 For each n-dimensional submanifold $i : N \to M$, we have a map $i^{(k)} : N \to J^k_n(M)$, which sends each point $q \in N$ to the k-jet of N at q. It is a smooth inmersion, and it verifies $\pi_{k,k-1} i^{(k)} = i^{(k-1)}$. This defines a submanifold $N^{(k)}$ of $J^k_n(M)$ called the *k-th prolongation* of N. This is the canonical way of lifting N to a submanifold on the jet spaces. Of course, not every submanifold of $J^k_n(M)$ arises as the prolongation of a submanifold of M. A first condition is that the given submanifold should be transverse to the vertical distribution $V\pi_{k,k-1} = \ker \pi_{k,k-1*}$. Moreover, it turns out that for every N the prolongation $N^{(k)}$ is an integral manifold of a certain distribution on $J^k_n(M)$, called the contact distribution. This two properties completely characterize the prolongations $N^{(k)}$ among all submanifolds of $J^k_n(M)$, as we will see later in this chapter.

We may iterate this construction and consider the l-th prolongation of $N^{(k)}$, which is now a submanifold $N^{(k)(l)}$ of $J^l_n(J^k_n(M))$. For each $q \in N$ the point $i^{(k)(l)}(q) \in J^l_n(J^k_n(M))$ only depends on the $(k+l)$-th jet of N at q, as can be seen in coordinates. Thus we get a mapping $J^{k+l}_n(M) \to J^l_n(J^k_n(M))$, which is in fact an embedding. It is not surjective; its image may be characterized as a prolongation of the contact system on $J^k_n(M)$. We will go back to this in chapter III.

1.4 Each $y \in J^1_n(M)$ defines a subspace $U_y \subseteq T_{y_0}M$, where $y_0 = \pi_{1,0}(y)$. The map $y \mapsto U_y$ defines a subbundle $U \subseteq \pi^*_{1,0}TM$ called the *universal bundle* on $J^1_n(M)$. From now on, we implicitly pullback bundles over $J^j_n(M)$ to $J^k_n(M)$ for $k > j$, as the base will be clear by context, so we may simply say $U \subseteq TM$. Let $Q = TM/U$.

More generally, we may pullback the universal bundle on $J^1_n(J^k_n(M))$ to $J^{k+1}_n(M)$ via the canonical embedding $J^{k+1}_n(M) \subseteq J^1_n(J^k_n(M))$, and we get a subbundle $U^{(k)} \subseteq TJ^k_n(M)$ defined over $J^{k+1}_n(M)$, which is called the *k-th prolongation* of the universal bundle. Observe that $\pi_{k,k-1*} : TJ^k_n(M) \to TJ^{k-1}_n(M)$ restricts to give an isomorphism $U^{(k)} \to U^{(k-1)}$, and so all the prolongations are isomorphic as bundles. We use the notation $U^{(k)}$ if we want to emphasize its embedding as a subbundle of $TJ^k_n(M)$. When we are only interested in the bundle structure we may identify them and simply speak of U. In particular, we usually do not need to distinguish the dual bundles $U^{(k)*}$, so we simply speak of U^*.

In coordinates, we have

$$U^{(k-1)} = \operatorname{span}\left\{\frac{\partial}{\partial x^i} + \sum_{0\le|I|\le k-1} u^a_{Ii}\frac{\partial}{\partial u^a_I}\right\} \tag{1}$$

$$U^* = \operatorname{span}\left\{\mathrm{d}x^i\right\} \tag{2}$$

$$Q = \operatorname{span}\left\{\frac{\partial}{\partial u^a}\right\} \tag{3}$$

1.5 We now want to study the bundles $J^k_n(M) \to J^{k-1}_n(M)$. When $k = 1$, we already know that $J^1_n(M)$ is a bundle of Grassmannians. Its vertical distribution may be described as follows

Proposition 1.5.1. *Let $V\pi_{1,0} = \ker \pi_{1,0*}$ be the vertical distribution on $J^1_n(M)$. Then $V\pi_{1,0} = U^* \otimes Q$.*

Proof. Let $q \in M$. The fiber $\pi^{-1}_{1,0}(q)$ is the Grassmannian of n-dimensional subspaces of T_qM. This is an homogeneous space for the group $GL(T_qM)$. If y is an element of the fiber, then the isotropy of the action at y is the subgroup H of automorphisms fixing U_y. The Lie algebra of H is the subalgebra $\mathfrak{h} \subseteq End(T_qM)$ of endomorphisms fixing U_y. The tangent space to the fiber at y is then $End(T_yM)/\mathfrak{h} = Hom(U_y, Q_y) = U^*_y \otimes Q_y$. □

In coordinates, $V\pi_{1,0}$ is spanned by the vectors $\partial/\partial u^a_i$, and the isomorphism maps $\partial/\partial u^a_i$ to $\mathrm{d}x^i \otimes \partial/\partial u^a$.

There is an alternative way of giving that isomorphism. Let γ be a smooth, vertical curve on $J^1_n(M)$, such that $\gamma(0) = y$. Let α be a smooth curve in T_qM such that $\alpha(t) \in \gamma(t)$ for all t. Then the contraction of $\dot\gamma(0)$ with $\alpha(0)$ is the projection of $\dot\alpha(0)$ to Q_y.

We now examine the case $k \ge 2$. Here we recover the intuitive fact that two k-jets differ by an homogeneous polynomial of order k.

Proposition 1.5.2. *The projection $\pi_{k,k-1} : J^k_n(M) \to J^{k-1}_n(M)$ is an affine bundle modeled on $S^kU^* \otimes Q$ for $k \ge 2$. In particular, $V\pi_{k,k-1} = S^kU^* \otimes Q$ for $k \ge 1$.*

Proof. Let $k \ge 2$ and assume that the proposition has been proven for $j < k$. For each $y \in J^{k-1}_n(M)$, let $\check{J}^k_n(M)_y$ be the space of lifts of $U^{(k-2)}_y$ to $T_yJ^{k-1}_n(M)$. This is the same as the splittings of

$$0 \to V_y\pi_{k-1,k-2} \to \pi^{-1}_{k-1,k-2*}(U^{(k-2)}_y) \to U^{(k-2)}_y \to 0 \tag{4}$$

and therefore $\check{J}^k_n(M)_y$ is an affine space modeled on $U^*_y \otimes V_y\pi_{k-1,k-2} = U^*_y \otimes (S^{k-1}U^*_y \otimes Q_y)$. The spaces $\check{J}^k_n(M)_y$ form an affine bundle $\check{J}^k_n(M)$ over $J^{k-1}_n(M)$ which embeds in

$J^1_n(J^{k-1}_n(M))$, called the space of k-th order sesqui-holonomic jets. The image of the canonical embedding of $J^k_n(M)$ inside $J^1_n(J^{k-1}_n(M))$ is actually contained in $\check{J}^k_n(M)$. We claim that $J^k_n(M)$ it is an affine subbundle of $\check{J}^k_n(M)$ modeled on $S^kU^* \otimes Q$.

To see that, we work locally. Let $w, \bar{w}$ be elements in the fiber of $\pi_{k,k-1}$ over y. Let (x^i, u^a_I) and $(x^i, \bar{u}^a_I)$ be their coordinates, with $u^a_I = \bar{u}^a_I$ for $|I| < k$. Then

$$\bar{w} - w = (\bar{u}^a_{Ii} - u^a_{Ii})\mathrm{d}x^i \otimes \partial/\partial u^a_I \in U^*_y \otimes V_y\pi_{k-1,k-2} \tag{5}$$

By induction, we may assume that this corresponds to

$$(\bar{u}^a_{Ii} - u^a_{Ii})\mathrm{d}x^i \otimes (\mathrm{d}x^I \otimes \partial/\partial u^a) \in U^*_y \otimes (S^{k-1}U^*_y \otimes Q_y) \tag{6}$$

Using the canonical embedding $S^kU^*_y \otimes Q_y$ inside $U^*_y \otimes (S^{k-1}U^*_y \otimes Q_y)$, this corresponds to $(\bar{u}^a_I - u^a_I)\mathrm{d}x^I \otimes \partial/\partial u^a$. Therefore, the possible differences of two elements in the fiber are in correspondence with $S^kU^*_y \otimes Q_y$, and so $J^k_n(M)$ is an affine bundle modeled on $S^kU^* \otimes Q$. □

1.6 Let M, M' be two differentiable manifolds, and $n \leq \dim M, \dim M'$. Let $\varphi : M \to M'$ be a smooth map. Let V be the open subspace of $J^1_n(M)$ consisting of those 1-jets y such that $\varphi_*|_{U_y}$ is a monomorphism. Let $i : N \to M$ be an n-dimensional submanifold passing through a point $q \in M$, such that $N^{(1)}$ is contained in V. Then $\varphi(N)$ is an n-dimensional submanifold of M', and the k-jet of $\varphi(N)$ at $\varphi(q)$ only depends on the k-jet of N at q. Therefore, we get a well defined map $\varphi^{(k)} : \pi^{-1}_{k,1}(V) \to J^k_n(M')$ called the *k-th prolongation* of φ. We have already seen a particular case of this: if $i : N \to M$ is an n-dimensional submanifold, then the k-th prolongation of the inclusion is the map $i^{(k)} : J^k_n(N) = N \to J^k_n(M)$ previously defined. Observe that if $\psi : M' \to M''$ is a smooth map then $(\psi\varphi)^{(k)} = \psi^{(k)}\varphi^{(k)}$ where it makes sense. Moreover, $\varphi^{(k+l)}$ coincides with the restriction of the iterated prolongation $\varphi^{(k)(l)}$ to $J^{k+l}_n(M) \subseteq J^l_n(J^k_n(M))$.

Let U' be the universal bundle on $J^1_n(M')$, and let $U'^{(k)}$ be its k-th prolongation. Let $Q' = TM'/U'$. We denote by $\pi_{k,j}$ both the projection $J^k_n(M) \to J^j_n(M')$ and the projection $J^k_n(M') \to J^j_n(M')$. Observe that the prolongations of φ commute with the projections. We implicitly pullback bundles on $J^k_n(M')$ to $\pi^{-1}_{k,1}(V)$, using $\varphi^{(k)}$. Observe that $\varphi^{(k)}_*$ gives an isomorphism between $U^{(k)}$ and $U'^{(k)}$ over $J^{k+1}_n(M)$. Therefore, unless we are interested in the embedding as a subbundle of a particular jet space, we may identify all the universal bundles and simply speak of U.

Let $\sigma_\varphi : Q \to Q'$ be the map induced by φ_*. It is called the *symbol* of φ. For $k \geq 0$,the *k-th prolongation* of the symbol of φ is the map

$$\sigma^{(k)}_\varphi = 1_{S^kU^*} \otimes \sigma_\varphi : S^kU^* \otimes Q \to S^kU^* \otimes Q' \tag{7}$$

The prolonged symbols determine the behavior of the prolongations of φ. We first prove this for the first prolongation

Lemma 1.6.1. *The map $U^* \otimes Q \to U^* \otimes Q'$ induced by restriction of $\varphi^{(1)}$ to the fibers of $\pi_{1,0}$ coincides with $\sigma_\varphi^{(1)}$.*

Proof. Let $y \in V$, and let γ be a smooth vertical curve inside V such that $\gamma(0) = y$. Let α be a smooth curve in $T_{y_0}M$ such that $\alpha(t) \in \gamma(t)$ for all t. Then the contraction of $\dot\gamma(0)$ with $\alpha(0)$ is given by the projection of $\dot\alpha(0)$ to Q_y. In the same way, the contraction of $\partial_t(\varphi^{(1)}\gamma)(0)$ with $\varphi_*\alpha(0)$ is given by $\partial_t(\varphi_*\alpha)(0)$ projected to $Q'_{\varphi^{(1)}y}$. The proposition then follows from the fact that $\partial_t(\varphi_*\alpha)(0) = \varphi_*\dot\alpha(0)$. □

In general, one has the following

Proposition 1.6.2. *The map $\varphi^{(k)} : \pi_{k,1}^{-1}(V) \to J_n^k(M')$ is an affine bundle morphism over $\varphi^{(k-1)} : \pi_{k-1,1}^{-1}(V) \to J_n^{k-1}(M')$ for $k \geq 2$, with associated vector bundle map $\sigma_\varphi^{(k)}$. In particular, the map $S^kU^* \otimes Q \to S^kU^* \otimes Q'$ induced by the restriction of $\varphi_*^{(k)}$ to the fibers of $\pi_{k,k-1}$ coincides with $\sigma_\varphi^{(k)}$ for all $k \geq 1$.*

Proof. Let $k \geq 2$, and suppose that we have already proven the result for all $j < k$. Let $y, \bar y$ be two k-jets inside $\pi_{k,1}^{-1}(V)$, such that $y_{k-1} = \bar y_{k-1}$. Let $y' = \varphi^{(k)}(y)$ and $\bar y' = \varphi^{(k)}(\bar y)$. We know that $U_y^{(k-1)}$ and $U_{\bar y}^{(k-1)}$ are two splittings at y_{k-1} of

$$0 \to V\pi_{k-1,k-2} \to \pi_{k-1,k-2*}^{-1}(U^{(k-2)}) \to U^{(k-2)} \to 0 \tag{8}$$

Their difference is, then, an element $\Delta \in U_y^* \otimes V_{y_{k-1}}\pi_{k-1,k-2}$. In the same way, $U'^{(k-1)}_{y'}$ and $U'^{(k-1)}_{\bar y'}$ are two splittings at y'_{k-1} of

$$0 \to V\pi_{k-1,k-2} \to \pi_{k-1,k-2*}^{-1}(U'^{(k-2)}) \to U'^{(k-2)} \to 0 \tag{9}$$

Their difference is an element $\Delta' \in U_y^* \otimes V_{y'_{k-1}}\pi_{k-1,k-2}$. Using the inductive hypothesis, we have,

$$\mathrm{id}_{U^*} \otimes \sigma_\varphi^{(k-1)}(\Delta) = \Delta' \tag{10}$$

It follows that $y' - \bar y' = \sigma_\varphi^{(k)}(y - \bar y)$, as we wanted. □

2 Jets of Sections

2.1 Let M be a manifold and $\xi : E_\xi \to M$ be a fibered manifold over M (that is, a surjective submersion). Fix $n \leq \dim M$. For $k \geq 1$, we denote by $J_n^k(\xi)$ the open submanifold of $J_n^k(E_\xi)$ given by those jets of submanifolds transverse to the vertical of ξ. We also set $J_n^0(\xi) = E_\xi$.

The space $J_n^k(\xi)$ is called the space of *k-jets of sections* of ξ along n-dimensional submanifolds of M. An element of $J_n^k(\xi)$ over $x \in M$ may be (non uniquely) represented by a pair (N, s), where N is an n-dimensional submanifold of M passing through x, and s is a section of $\xi|_N$.

The prolongations of ξ are maps $\xi^{(k)} : J^k_n(\xi) \to J^k_n(M)$ giving $J^k_n(\xi)$ the structure of a fibered manifold over $J^k_n(M)$. Given a section $s : M \to E_\xi$, its k-th prolongation $s^{(k)} : J^k_n(M) \to J^k_n(\xi)$ is a section of $\xi^{(k)}$. Given a morphism of fibered manifolds $\varphi : E_\xi \to E_\eta$ over M, its k-th prolongation is a morphism of fibered manifolds $\varphi^{(k)} : J^k_n(\xi) \to J^k_n(\eta)$ over $J^k_n(M)$. Prolongation is compatible with pullbacks: if $\varphi : M' \to M$ is a smooth map, then $(\varphi^*\xi)^{(k)}$ is the pullback of $\xi^{(k)}$ over $\varphi^{(k)}$.

Let x^i, u^a be local coordinates on M, and choose functions v^b on E_ξ such that x^i, u^a, v^b is a system of coordinates on E_ξ. This induces a coordinate system x^i, u^a_I, v^b_J on $J^k_n(E_\xi)$. The domain of definition of this coordinate system is contained in $J^k_n(\xi)$, so they form a coordinate system for this manifold. Moreover, any point of $J^k_n(\xi)$ belongs to one of these domains, so these coordinate systems form an atlas for $J^k_n(\xi)$. The map $\xi^{(k)} : J^k_n(\xi) \to J^k_n(M)$ is given locally by forgetting the coordinates v^b_J. If $s : M \to E_\xi$ is given in coordinates by $v^b = s^b(x^i, u^a)$, then $s^{(k)}$ is given by $v^b_J = D_J s^b(x^i, u^a_I)$, where $D_J s^b$ denotes the derivative with respect to $x^{J_1}, \ldots, x^{J_n}$, treating u^a as functions of the variables x^i (with derivatives u^a_J).

2.2 An important particular case of the theory in this section is when $n = \dim M$. In this case we drop the n and use the notation $J^k(\xi)$. It is the space of *k-th jets of sections* of ξ. Two sections represent the same k-th jet at $x \in M$ if, on any coordinate system, their derivatives agree up through order k. An even more particular case of this (which is still very important) is the case of a trivial bundle $M \times F \to M$. Here sections are in correspondence with smooth functions from M to F, and so the theory that we develop naturally includes the theory of differential equations and differential operators on functions.

When $n = \dim M$, we have that $J^k_n(M) = M$ for all k, so the spaces $J^k(\xi)$ should all be considered over the same base M. In this case, the universal bundle on $J^k(\xi)$ may be considered as a connection on $\xi^{(k)} : J^k(\xi) \to M$, which is only well defined when we pass to $J^{k+1}(\xi)$.

Even if one is only interested in this case, when linearising one needs to consider the space $J^k_n(V\xi \to E_\xi)$, and so spaces of sections over (non necessarily top dimensional) submanifolds necessarily turn up (the exception being when the fibers of ξ are paralelizable: here $V\xi$ is actually defined over M, and so one stays in the realm of sections over open submanifolds).

2.3 When $n < \dim M$, the spaces $J^k_n(\xi)$ are fibered manifolds over different base manifolds. As usual, we shall implicitly pullback $\xi^{(k)}$ along $\pi_{k+l,k} : J^{k+l}_n(M) \to J^k_n(M)$ for $l \geq 0$. For example, if we speak of the map $J^{k+1}_n(\xi) \to J^k_n(\xi)$, we shall assume that $J^k_n(\xi)$ has been pullbacked over $\pi_{k+1,k}$, so that we are dealing with a morphism of fibered manifolds over $J^{k+1}_n(M)$.

Some results from the first section will have to be adapted to be consistent with this philosophy. For example, we already know that the fibers of $J^{k+1}_n(E_\xi) \to J^k_n(E_\xi)$ are affine spaces modeled on a certain bundle of polynomials. However, this does not directly apply to the fibers of $J^{k+1}_n(\xi) \to J^k_n(\xi)$ since one has to take into account the fact that $J^k_n(\xi)$ has been pullbacked.

As a general rule, when $n = \dim M$ all the results and definitions from the theory of jets of submanifolds apply directly, and in the case $n < \dim M$ definitions have to be modified in order to be compatible with pullbacks $M' \to M$.

For example, there is no problem with the vertical bundle $V\xi$: this is independent of the base of the fibered manifold, and only depends on the fibers. However, the tangent bundle TE_ξ depends on the base manifold M. The correct analogue of the tangent bundle TE_ξ when working with sections over n-dimensional submanifolds is $T_n(E_\xi) = \xi_*^{-1}(U)$, where U is the universal bundle on $J^1_n(M)$. This is a bundle defined over the pullback of E_ξ to $J^1_n(M)$. It is obviously compatible with pullbacks, and in the case $n = \dim M$ we are left with $\xi_*^{-1}(TM) = TE_\xi$. After one adapts all the concepts to be compatible with this rule, the general theory is indistinguishable from the case $n = \dim M$. Indeed, most results may be proven by restricting to this case, after pullbacking ξ to an n-dimensional submanifold of M.

2.4 The universal bundle on $J^k_n(E_\xi)$ restricts to give a subbundle $U^{(k-1)}_\xi \subseteq T_n J^{k-1}_n(\xi)$. Set $U_\xi = U^{(0)}_\xi$. Observe that we have an exact sequence

$$0 \to U_\xi \to T_n E_\xi \to V\xi \to 0 \tag{11}$$

and so the correct analogue of the universal quotient bundle is $V\xi$. Of course, $\xi^{(k)}$ induces isomorphisms between the bundles $U^{(k)}$ and $U^{(k)}_\xi$, so we will sometimes identify them and simply speak of U.

If $\varphi : E_\xi \to E_\eta$ is a morphism of fibered manifolds, we define its symbol as $\sigma_\varphi = \varphi_*|_{V\xi} : V\xi \to V\eta$. The k-th prolongation of the symbol is defined as

$$\sigma^{(k)}_\varphi = 1_{S^k U^*} \otimes \sigma_\varphi : S^k U^* \otimes V\xi \to S^k U^* \otimes V\eta \tag{12}$$

The analogue of propositions 1.5.2 and 1.6.2 in this context is

Proposition 2.4.1. *Let $k \geq 1$. The map $J^k_n(\xi) \to J^{k-1}_n(\xi)$ is an affine bundle modeled on $S^k U^* \otimes V\xi$. Moreover, if $\varphi : E_\xi \to E_\eta$ is a morphism of fibered manifolds, then $\varphi^{(k)} : J^k_n(\xi) \to J^k_n(\eta)$ is a morphism of affine bundles over $\varphi^{(k-1)}$, with associated vector bundle map $\sigma^{(k)}_\varphi$.*

Proof. Consider first the case $k = 1$. Let $q \in M$, and let $y \in \pi^{-1}_{1,0}(q)$ and $e \in \xi^{-1}(x)$. The fiber of $J^1_n(\xi)$ over (y, e) is in correspondence with the splittings of

$$0 \to V_e\xi \to \xi_*^{-1}(U_y) \xrightarrow{\xi_*} U_y \to 0 \tag{13}$$

and therefore it is an affine space modeled on $U_y^* \otimes V_e\xi$. This gives the desired affine bundle structure on $J^1_n(\xi)$.

We now let $k \geq 2$. Consider the following commutative diagram

$$\begin{array}{ccc} J^k_n(\xi) & \xrightarrow{\xi^{(k)}} & J^k_n(M) \\ \downarrow & & \downarrow \\ J^{k-1}_n(\xi) & \xrightarrow{\xi^{(k-1)}} & J^{k-1}_n(M) \end{array} \tag{14}$$

We know that $\xi^{(k)}$ is a morphism of affine bundles over $\xi^{(k-1)}$, with associated vector bundle map $\sigma^{(k)}_\xi$. We now pullback this diagram to lie over $J^k_n(M)$. In order to avoid confusion, we shall make the pullbacks explicit

$$\begin{array}{ccc} J^k_n(\xi) \times_{J^{k-1}_n(M)} J^k_n(M) & \xrightarrow{\xi^{(k)} \times 1_{J^k_n(M)}} & J^k_n(M) \times_{J^{k-1}_n(M)} J^k_n(M) \\ \downarrow & & \downarrow \\ J^{k-1}_n(\xi) \times_{J^{k-1}_n(M)} J^k_n(M) & \xrightarrow{\xi^{(k-1)} \times 1_{J^k_n(M)}} & J^k_n(M) \end{array} \tag{15}$$

The vertical arrows are still affine bundles, and the horizontal arrows define an affine bundle map. There is now a canonical section of the right vertical arrow, so this is now a vector bundle. The bundle

$$J^k_n(\xi) \to J^{k-1}_n(\xi) \times_{J^{k-1}_n(M)} J^k_n(M) \tag{16}$$

is obtained as the kernel of $\xi^{(k)} \times 1_{J^k_n(M)}$, so it is an affine bundle modeled on $\ker \sigma^{(k)}_\xi = S^kU^* \otimes \ker \sigma_\xi = S^kU^* \otimes V\xi$, as we wanted

The fact that the prolongation $\varphi^{(k)}$ of a fibered map φ is an affine bundle morphism with associated vector bundle map $\sigma^{(k)}_\varphi$ follows from proposition 1.6.2 in the case $k \geq 2$. We leave the case $k = 1$ for the reader. □

In coordinates, two points $w, \bar{w}$ in the same fiber of $J^k_n(\xi) \to J^{k-1}_n(\xi)$ may be written as (x^i, u^a_I, v^b_I) and $(x^i, u^a_I, \bar{v}^b_I)$, where $v^b_I = \bar{v}^b_I$ unless $|I| = k$. Their difference $\bar{w} - w \in S^kU^* \otimes V\xi$ is then $(\bar{v}^b_I - v^b_I)\mathrm{d}x^I \otimes \partial/\partial v^b$.

2.5 If $\xi : E_\xi \to M$ is a vector bundle, then the prolongations $\xi^{(k)} : J^k_n(\xi) \to J^k_n(M)$ have vector bundle structures defined as follows. Let $z, \bar{z}$ be two points in the fiber of $y \in J^k_n(M)_x$, represented by the pairs (N, s) and $(N, \bar{s})$. Then $z + \bar{z}$ is given by the class of $(N, s + \bar{s})$, and, for $\lambda \in \mathbb{R}$ a scalar, λy is given by the class of $(N, \lambda s)$. Prolongation respects sends vector bundle morphisms to vector bundle morphisms, and respects exact sequences.

Observe that $\pi_{k,k-1} : J^k_n(\xi) \to J^k_n(\xi)$ is a vector bundle map. Proposition 2.4.1 implies that $\ker(\pi_{k,k-1} : J^k_n(\xi) \to J^{k-1}_n(\xi)) = S^kU^* \otimes E_\xi$ as vector bundles over $J^k_n(M)$. Let $\varphi : E_\xi \to E_\eta$ be a morphism of vector bundles. Then the symbol σ_φ may be identified with φ. Moreover, $\varphi^{(k)}$ restricts to a morphism $S^kU^* \otimes E_\xi$ which coincides with $\sigma^{(k)}_\varphi$.

2.6 Let M be an n-dimensional manifold and fix $n \leq \dim M$. Let ξ_U, ξ_Q denote the projections from U and Q to $J^1_n(M)$, and ξ_{TM} denote the projection from TM to M. For each $k \geq 0$ we may consider $J^k_n(\xi_U)$ and $J^k_n(\xi_Q)$ as vector bundles over $J^{k+1}_n(M)$, via the canonical embedding $J^{k+1}_n(M) \subseteq J^k_n(J^1_n(M))$. The following proposition and its corollary will be useful when we discuss linearization of nonlinear differential operators and equations.

Proposition 2.6.1. *Let M be a differentiable manifold and $n \leq \dim M$. Let $k \geq 0$. There is a short exact sequence of bundles over $J^{k+1}_n(M)$*

$$0 \to \mathcal{H}^k \to J^k_n(\xi_{TM}) \to TJ^k_n(M) \to 0 \tag{17}$$

where $\mathcal{H}^k$ is the kernel of the canonical map $J^k_n(\xi_U) \to U$.

Proof. We begin by constructing the map $J^k_n(\xi_{TM}) \to TJ^k_n(M)$. Let $i : N \to M$ be an n-dimensional submanifold of M passing through a point q, and s be a section of $TM|_N$. Let $i_t : N \to M$ be a one-parameter family of immersions such that $i_0 = i$ and $s = \partial_t|_{t=0} i_t$. Then the tangent vector $\partial_t|_{t=0} i^{(k)}_t(q)$ to $J^k_n(M)$ only depends on the k-th jet of (N, s), and so this defines a map $J^k_n(\xi_{TM}) \to TJ^k_n(M)$ over $J^k_n(M)$. It is easily seen that any tangent vector in $TJ^k_n(M)$ may be obtained from a family of immersions $i_t : N \to M$, so this map is surjective.

The inclusion $U \subseteq TM$ induces an inclusion $J^k_n(\xi_U) \subseteq J^k_n(\xi_M)$ over $J^{k+1}_n(M)$, which restricts to give the first map in the short exact sequence of the statement. Moreover, observe that the composition is zero. Indeed, an element of $\mathcal{H}^k \subseteq J^k_n(\xi_{TM})$ may be represented as a pair (N, s) where N is an n-dimensional submanifold of M passing through a point q, and s is a section of TN which vanishes at q. The variation i_t may be constructed so that $i_t(q)$ is constant, and $i_t(N) \subseteq i_0(N)$ near q. This implies that the k-jet of $i_t(N)$ at q remains constant, and therefore its derivative is zero.

It remains to prove exactness at $J^k_n(\xi_{TM})$. We proceed by induction. When $k = 0$, the sequence is simply

$$0 \to 0 \to TM \to TM \to 0 \tag{18}$$

which is obviously exact. The inductive step follows from looking at the following commutative diagram of bundles over $J^{k+1}_n(M)$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^kU^* \otimes U & \longrightarrow & S^kU^* \otimes TM & \longrightarrow & S^kU^* \otimes Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}^k & \longrightarrow & J^k_n(\xi_{TM}) & \longrightarrow & TJ^k_n(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}^{k-1} & \longrightarrow & J^{k-1}_n(\xi_{TM}) & \longrightarrow & TJ^{k-1}_n(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{19}
$$

The columns are exact and the rows are exact except possibly at $J^k_n(\xi_{TM})$. From this and some diagram chasing, we get that it must also be exact at $J^k_n(\xi_{TM})$, as we wanted. □

Corollary 2.6.2. *Let M be a differentiable manifold and $n \leq \dim M$. Let $k \geq 0$. There is an exact sequence of bundles over $J^{k+1}_n(M)$*

$$0 \to U^{(k)} \to TJ^k_n(M) \to J^k_n(\xi_Q) \to 0 \tag{20}$$

Proof. Prolongation preserves exact sequences, so we have the following exact sequence of bundles over $J^{k+1}_n(M)$

$$0 \to J^k_n(\xi_U) \to J^k_n(\xi_{TM}) \to J^k_n(\xi_Q) \to 0 \tag{21}$$

Taking the quotient of the first two bundles by $\mathcal{H}^k$, we get an exact sequence

$$0 \to U \to TJ^k_n(M) \to J^k_n(\xi_Q) \to 0 \tag{22}$$

We claim that the first map is the canonical embedding of U inside $TJ^k_n(M)$ (i.e., the k-th prolongation of U). To see this, let $y \in J^{k+1}_n(M)$ be a k-jet, and $v \in U_y$. Let w be an element in the fiber of $\xi^{(k)}_U$ over y such that its projection to U equals v. Then the image of w inside $J^k_n(\xi_{TM})$ may be represented as a pair (N, s) where N is an n-dimensional submanifold of M passing through y_0, and s is a section of TN such that $s(y_0) = v$. As before, from this we may construct a variation of N contained inside $i_0(N)$. The basepoint y_0 has initial speed v, and so $\partial_t|_{t=0} i^{(k)}_t(y_0)$ is the lift of v to $U^{(k)}_y$, which proves the claim.

□

The correct linearization result in the context of fibered manifolds is the following

PROPOSITION 2.6.3. *Let $\eta : E_\eta \to M$ be a fibered manifold. We denote by ξ_{U_η}, $\xi_{V\eta}$ and $\xi_{T_n E_\eta}$ the projections of the corresponding bundles. Let $k \geq 0$.*

1. *There is a short exact sequence of bundles over $J^{k+1}_n(\eta)$*

$$0 \to \mathcal{H}^k \to J^k_n(\xi_{T_n E_\eta}) \to T_n J^k_n(\eta) \to 0 \tag{23}$$

   where $\mathcal{H}^k$ is the kernel of the canonical map $J^k_n(\xi_{U_\eta}) \to U_\eta$.

2. *There is an isomorphism $V\eta^{(k)} = J^k_n(\xi_{V\eta})$ of bundles over $J^k_n(\eta)$.*

The exact sequence in the first item is contained inside the sequence that proposition 2.6.1 gives for E_ξ. To obtain the isomorphism in the second item one first restricts the sequence from corollary 2.6.2 and then uses the identity $T_n J^k_n(\eta)/U_\eta = V\eta^{(k)}$. We leave the details of the proof to the reader.

3 The Contact Distribution

3.1 Let M be a differentiable manifold and fix $n \leq \dim M$. The *contact distribution* on $J^k_n(M)$ is the subbundle $\mathcal{C}^k_n = \pi^{-1}_{k,k-1*}(U^{(k-1)}) \subseteq TJ^k_n(M)$. This sits in a short exact sequence

$$0 \to S^k U^* \otimes Q \to \mathcal{C}^k_n \to U^{(k-1)} \to 0 \tag{24}$$

The *bundle of contact forms* $I^{(k-1)} \subseteq T^* J^k_n(M)$ is the annihilator of $\mathcal{C}^k_n$. Using 2.6.2, we have

$$I^{(k-1)} = (TJ^k_n(M)/\mathcal{C}^k_n)^* = (TJ^{k-1}_n(M)/U^{(k-1)})^* = J^{k-1}_n(\xi_Q)^* \tag{25}$$

Observe that $I^{(0)} = Q^*$. Moreover, $I^{(k-2)} \subseteq I^{(k-1)}$ as subbundles of $T^*J^k_n(M)$, and $I^{(k-1)}/I^{(k-2)} = S^{k-1}U \otimes Q^*$.

In coordinates x^i, u^a_I on $J^k_n(M)$, the contact forms are spanned by the 1-forms $\mathrm{d}u^a_I - u^a_{Ii}\mathrm{d}x^i$, for $0 \leq |I| < k$. The quotient $I^{(k-1)}/I^{(k-2)}$ is spanned by the forms $\mathrm{d}u^a_I - u^a_{Ii}\mathrm{d}x^i$ for $|I| = k-1$, and the isomorphism with $S^{k-1}U \otimes Q^*$ is given by

$$\theta^a_I = \mathrm{d}u^a_I - u^a_{Ii}\mathrm{d}x^i \mapsto \partial_I \otimes \mathrm{d}u^a \tag{26}$$

where $\partial_I = \partial_{I_1} \dots \partial_{I_{k-1}}$, with ∂_i the basis of U dual to $\mathrm{d}x^i$.

The contact distribution allows us to determine which submanifolds of $J^k_n(M)$ arise as prolongations of submanifolds of M:

PROPOSITION 3.1.1. *Prolongation $N \mapsto N^{(k)}$ defines a correspondence between n-dimensional submanifolds of M and n-dimensional integral manifolds of $\mathcal{C}^k_n$ which are transverse to $V\pi_{k,k-1}$.*

Proof. Observe that if N is a submanifold of M, then $N^{(k)}$ is an integral manifold submanifold of $\mathcal{C}_n^k$ transverse to $V\pi_{k,k-1}$, and we recover N as $(\pi_{k,0}N)^{(k)}$. The only thing left to do is to show that if $\bar{N}$ is an n-dimensional integral manifold of $\mathcal{C}_n^k$ transverse to $V\pi_{k,k-1}$, then $\bar{N} = (\pi_{k,0}\bar{N})^{(k)}$.

We proceed by induction. The case $k = 1$ is immediate. Now, let $k \geq 2$. Applying the result for $\pi_{k,k-1}(\bar{N})$, we see that $\pi_{k,k-1}\bar{N} = (\pi_{k,0}\bar{N})^{(k-1)}$. The fact that $\bar{N}$ is integral for $\mathcal{C}_n^k$ implies that $\bar{N} = ((\pi_{k,0}\bar{N})^{(k-1)})^{(1)}$ inside of $J_n^1(J_n^{k-1}(M))$. Restricting to $J_n^k(M)$ we get the desired equality $\bar{N} = (\pi_{k,0}\bar{N})^{(k)}$. □

3.2 From the coordinate description we see that, unless $n = \dim M$, the contact distribution is not integrable. This means that the map

$$\overline{[\cdot,\cdot]} : \Lambda^2\mathcal{C}_n^k \to TJ_n^k(M)/\mathcal{C}_n^k = J_n^{k-1}(\xi_Q) \tag{27}$$

induced by the Lie bracket, is typically nonzero. Our next goal is to understand this morphism.

From (24) we get a filtration

$$F_0 \subset F_1 \subset F_2 = \Lambda^2(\mathcal{C}_n^k) \tag{28}$$

where $F_0 = \Lambda^2 V\pi_{k,k-1}$ and $F_1 = \operatorname{im}(V\pi_{k,k-1} \otimes \mathcal{C}_n^k \xrightarrow{\wedge} \Lambda^2\mathcal{C}_n^k)$. We have that $F_1/F_0 = V\pi_{k,k-1} \otimes U^{(k-1)}$ and $F_2/F_1 = \Lambda^2(U^{(k-1)})$, where the first isomorphism is induced by contraction with U^*, and the second is induced by the projection $\mathcal{C}_n^k \to U^{(k-1)}$.

The vertical distribution is integrable, and so $\overline{[\cdot,\cdot]}$ passes to the quotient $F_2/F_0 \to J_n^{k-1}(\xi_Q)$. Moreover, when working over $J_n^{k+1}(M)$, sequence (24) splits, and so we get a decomposition $\Lambda^2(\mathcal{C}_n^k) = F_0 \oplus (F_1/F_0) \oplus (F_2/F_1)$. Since for each $y \in J_n^{k+1}(M)$ the plane $U_y^{(k)}$ may be extended to an integral submanifold of $\mathcal{C}_n^k$, we have that $\overline{[\cdot,\cdot]}$ vanishes when restricted to $F_2/F_1 = \Lambda^2 U^{(k)}$. As a consequence of this, the image of $\overline{[\cdot,\cdot]}$ does not change when we restrict it to F_1/F_0. The following proposition describes this restriction:

Proposition 3.2.1. *The image of* $\overline{[\cdot,\cdot]}$ *is contained in* $S^{k-1}U^* \otimes Q \subseteq J_n^{k-1}(\xi_Q)$. *Moreover, the induced map*

$$F_1/F_0 = (S^k U^* \otimes Q) \otimes U \to S^{k-1}U^* \otimes Q \tag{29}$$

is the contraction mapping.

Proof. Let $y \in J_n^k(M)$, and X_y, Y_y be two tangent vectors at y, with $X_y \in V\pi_{k,k-1}$ and $Y_y \in \mathcal{C}_n^k$. We extend X_y to a vector field X tangent to $V\pi_{k,k-1}$. Let Z be a section of $U^{(k-2)}$ over $J_n^{k-1}(M)$ such that $Z_{y_{k-1}} = \pi_{k,k-2*}(Y_y)$. Extend Y_y to a vector field on $J_n^k(M)$, tangent to $\mathcal{C}_n^k$, such that $\pi_{k,k-2*}Y = Z$. Observe that the flow of X preserves the projection $\pi_{k,k-2}$, and $\pi_{k,k-2*}Y$ is constant along the flowlines of X. These two facts combined imply that $\pi_{k,k-2*}L_X Y = 0$. This shows that $\overline{[X,Y]} \in S^{k-1}U^* \otimes Q$, as we claimed.

To see that the induced map on F_1/F_0 is contraction, we work in local coordinates x^i, u^a_I on $J^k_n(M)$. The bundle U^* is spanned by $\mathrm{d}x^i$, and Q is spanned by (the classes of) $\partial/\partial u^a$. Let ∂_i be the basis of U dual to $\mathrm{d}x^i$. We want to compute $\overline{[\mathrm{d}x^I \otimes \partial/\partial u^a, \partial_i]}$. To do this, observe that $\mathrm{d}x^I \otimes \partial/\partial u^a$ corresponds to the vertical vector $\partial/\partial u^a_I$, and ∂_i may be lifted to the contact vector field $\partial/\partial x^i + u^b_{Ji}\partial/\partial u^b_J$. The Lie bracket of these two fields is $\partial/\partial u^a_J$, where J is the multi-index such that $Ji = I$. This is the vector corresponding to the contraction of $\mathrm{d}x^I \otimes u^a$ with ∂_i, which is what we wanted. $\square$

3.3 The above discussion on the Lie bracket of contact vector fields may be dualised to give us information about the exterior derivative of contact forms. Concretely, we are interested in the map

$$\delta : I^{(k-1)} \to \Lambda^2\mathcal{C}^{k*}_n \tag{30}$$

induced by exterior differentiation, which coincides with $-\overline{[\cdot,\cdot]}^*$. The dual to sequence (24) induces a filtration

$$F^0 \subset F^1 \subset F^2 = \Lambda^2\mathcal{C}^{k*}_n \tag{31}$$

where $F^0 = \Lambda^2 U^*$ and $F^1 = \mathrm{im}(U^* \otimes \mathcal{C}^{k*}_n \xrightarrow{\wedge} \Lambda^2\mathcal{C}^{k*}_n)$. We have that $F^1/F^0 = U^* \otimes V^*\pi_{k,k-1}$ and $F^2/F^1 = \Lambda^2 V^*\pi_{k,k-1}$. Here F^0 is the annihilator of F_1 and F^1 is the annihilator of F_0. Therefore, the image of δ is contained in F^1. Proposition 3.2.1 dualises to give

Proposition 3.3.1. *The map δ factors through $I^{(k-1)}/I^{(k-2)}$. The induced map*

$$\bar{\delta} : S^{k-1}U \otimes Q^* \to F^1/F^0 = U^* \otimes (S^k U \otimes Q^*) \tag{32}$$

is the map induced by multiplication $U \otimes (S^{k-1}U \otimes Q^*) \to S^k U \otimes Q^*$.

Of course, we could have seen this directly using local coordinates, without appealing to proposition 3.2.1. We have

$$F^0 = \mathrm{span}\{\mathrm{d}x^i \wedge \mathrm{d}x^j\} \tag{33}$$

$$F^1 = \mathrm{span}\{\mathrm{d}x^i \wedge \mathrm{d}x^j, \mathrm{d}x^i \wedge \mathrm{d}u^a_I\} \tag{34}$$

Let $\theta^a_I = \mathrm{d}u^a_I - u^a_{Ii}\mathrm{d}x^i$, for $|I| \le k-1$. The fact that $\mathrm{d}\theta^a_I = \mathrm{d}x^i \wedge \theta^a_{Ii}$ for $|I| \le k-2$ implies that δ vanishes when restricted to $I^{(k-2)}$. Moreover, the formula $\mathrm{d}\theta^a_I = \mathrm{d}x^i \wedge \mathrm{d}u^a_{Ii}$ shows that the image of δ is contained in F^1, and that the induced map $\bar{\delta}$ is given as in the above proposition.

3.4 We now discuss a kind of infinitesimal analogue to the correspondence in proposition 3.1.1. Let $y \in J^k_n(M)$. We want to characterize the subspaces of $T_y J^k_n(M)$ of the form $U^{(k)}_z$, for z in the fiber of $\pi_{k+1,k}$ over y. Of course, they are contained in $\mathcal{C}^k_{n,y}$, and are transverse to $V_y\pi_{k,k-1}$. Moreover, as we observed before, the form $\overline{[\cdot,\cdot]}$ vanishes when restricted to these planes, since they may be extended to integral submanifolds of $\mathcal{C}^k_n$.

This motivates the following definition: an n-dimensional subspace $\Pi \subseteq T_y J^k_n(M)$ is called an *integral element for the contact distribution* if it is contained in $\mathcal{C}^k_{n,y}$, the projection $\pi_{k,k-1*}$ restricts to give an isomorphism $\Pi = U^{(k-1)}_y$, and the form $\overline{[\cdot,\cdot]}_y$ vanishes when restricted to Π. Equivalently, Π is called an integral element of the contact distribution if for every $\theta \in I^{(k-1)}_y$ we have that both $\theta|_\Pi$ and $\delta\theta|_\Pi$ vanish, and moreover the induced map $U^* \to \Pi^*$ is an isomorphism.

It turns out that any integral element is of the form $U^{(k)}_z$ for some z:

PROPOSITION 3.4.1. *The image of the canonical embedding $J^{k+1}_n(M) \subseteq J^1_n(J^k_n(M))$ consists of the integral elements of the contact distribution on $J^k_n(M)$.*

Proof. We have already observed that $J^{k+1}_n(M)$ is contained inside the bundle of contact elements, so we only have to prove the other inclusion. Observe that the bundle of contact elements sits inside the bundle $\check{J}^{k+1}_n(M)$ of sesqui-holonomic jets, (see the proof of proposition 1.5.2 for the definition of this space). Therefore, to show that it equals $J^k_n(M)$ it suffices to see that the difference between two contact elements elements $\Pi, \bar{\Pi}$ at a point $y \in J^k_n(M)$ belongs to $S^{k+1}U^* \otimes Q$.

Let $X, Y \in U_y$, and let X_Π, Y_Π be their lifts to Π. Then their lifts to $\bar{\Pi}$ are $X_\Pi + \Delta(X)$ and $Y_\Pi + \Delta(Y)$, where $\Delta = \bar{\Pi} - \Pi \in U^* \otimes V\pi_{k,k-1} = U^* \otimes (S^{k-1}U^* \otimes Q)$. Using that Π and $\bar{\Pi}$ are integral elements for the contact distribution, we have

$$0 = \overline{[X_\Pi + \Delta(X), Y_\Pi + \Delta(Y)]} = \overline{[\Delta(X), Y_\Pi]} - \overline{[\Delta(Y), X_\Pi]} \tag{35}$$

By 3.2.1, this equals $Y \lrcorner \Delta(X) - X \lrcorner \Delta(Y)$. The fact that this vanishes for all X, Y implies that Δ is symmetric, which finishes the proof. □

In particular, we recover $J^{k+1}_n(M)$ from the knowledge of the space $J^k_n(M)$, the contact distribution $\mathcal{C}^k_n$ and the vertical distribution $V\pi_{k,k-1}$, as the submanifold of integral elements inside $J^1_n(J^k_n(M))$. This embedding also provides us with the bundle $U^{(k)}$, and therefore we may also recover $\mathcal{C}^{k+1}_n$. Moreover, $V\pi_{k+1,k}$ is simply the restriction to $J^{k+1}_n(M)$ of the vertical of the projection $J^1_n(J^k_n(M)) \to J^k_n(M)$. In this way, one could develop the theory of jet spaces in an inductive way. This is the viewpoint often taken in the literature on exterior differential systems.

Chapter II

Spencer Cohomology

This chapter contains the algebraic background needed throughout the rest of the thesis.

We begin in section 1 by discussing the Koszul homology of graded modules over a polynomial algebra. The polynomial Poincaré lemma implies that the Koszul homology may be computed in terms of a certain de Rham like complex, called the Koszul complex. We shall also give another description using a minimal resolution of the module; the equivalence of both descriptions depends on the commutativity of Tor.

Section 2 deals with the dual construction: the Spencer cohomology of graded comodules over a polynomial coalgebra. The concept of a tableau is introduced, which provides a common way of constructing comodules. We finish this section by computing the cohomology of some basic first order tableaux.

Section 3 is an introduction to the algebraic theory of involution. We begin by recalling the relationship between the existence of regular sequences and the vanishing of the Koszul homology (and dually, of the Spencer cohomology). Cartan's test is proven, which gives a practical way of determining if a tableau is involutive. We then give a normal form for first order tableaux in coordinates, and discuss the consequences of involutivity on this normal form.

This theory will be applied in chapter III: to any (sufficiently regular) differential equation there is an associated bundle of graded comodules over a certain polynomial coalgebra, constructed from the symbol of the equation. It turns out that the obstructions to the integrability of the equation live inside the second Spencer cohomology group of these comodules. Moreover, the involutivity of the equation is equivalent to the involutivity of the associated comodule.

The fact that for any finitely generated comodule the Spencer cohomology vanishes in sufficiently high degrees (which is a simple consequence of the Hilbert basis theorem) implies that certain prolongation processes terminate. This will be used in chapter III to prove a weak version of the Cartan-Kuranishi prolongation theorem, and again in chapter IV to prove that Cartan's method for solving the equivalence problem terminates.

1 Koszul Homology

1.1 Throughout this section, V is an n-dimensional vector space over a field F of characteristic zero. We denote by SV its symmetric algebra, and set $S^+V = \bigoplus_{k>0} S^kV$. We say that a graded SV module $\mathcal{M} = \oplus_{k\in\mathbb{Z}}\mathcal{M}^k$ is *quasi-finitely generated* (QFG for short) if the graded pieces $\mathcal{M}^k$ are finite dimensional vector spaces. Let $\mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$ be the (abelian) category of QFG non-negatively graded SV modules and degree-preserving morphisms.

Let $\mathcal{M} \in \mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$. We say that a free resolution

$$\ldots \to \mathcal{F}_2 \xrightarrow{\psi_2} \mathcal{F}_1 \xrightarrow{\psi_1} \mathcal{F}_0 \xrightarrow{\epsilon} \mathcal{M} \to 0 \tag{1}$$

of $\mathcal{M}$ in $\mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$ is *minimal* if $\operatorname{im}\psi_q \subseteq S^+V\mathcal{F}_{q-1}$ for all $q > 0$ (where $S^+V\mathcal{F}_{q-1}$ denotes the submodule of $\mathcal{F}_{q-1}$ spanned by the elements of the form Px with $P \in S^+V$ and $x \in \mathcal{F}_{q-1}$). Equivalently, the resolution is minimal if $\ker\psi_q \subseteq S^+V\mathcal{F}_q$ for all $q \geq 0$.

Proposition 1.1.1. *Let $\mathcal{M}$ be a QFG non-negatively graded SV module. There exists a unique minimal resolution of $\mathcal{M}$ up to (non unique) isomorphism.*

Proof. We may construct a minimal resolution of $\mathcal{M}$ as follows. Let $\mathcal{Q}$ be a complement for $S^+V\mathcal{M} \subseteq \mathcal{M}$ as graded vector spaces. Set $\mathcal{F}_0 = SV \otimes_{\mathbb{R}} \mathcal{Q}$. Multiplication induces an epimorphism $\epsilon : \mathcal{F}_0 \to \mathcal{M}$. Let $\mathcal{K}_0$ be its kernel and let $\mathcal{Q}_0$ be a complement for $S^+\mathcal{K}_0 \subseteq \mathcal{K}_0$ as graded vector spaces. Set $\mathcal{F}_1 = SV \otimes_{\mathbb{R}} \mathcal{Q}_0$. Multiplication induces an epimorphism $\psi_1 : \mathcal{F}_1 \to \mathcal{K}_0$. Continuing in this way, one gets a resolution

$$\ldots \to \mathcal{F}_2 \xrightarrow{\psi_2} \mathcal{F}_1 \xrightarrow{\psi_1} \mathcal{F}_0 \xrightarrow{\epsilon} \mathcal{M} \to 0 \tag{2}$$

of $\mathcal{M}$ by free QFG SV modules, which is easily seen to be minimal

Let $\mathcal{F}'_*$ be another minimal resolution. We will prove that there exists an isomorphism $\mathcal{F}_0 \simeq \mathcal{F}'_0$ which commutes with the augmentation morphisms, where $\mathcal{F}_*$ is a fixed resolution constructed as above (which depends choices of complements $\mathcal{Q}$ for $S^+V\mathcal{M}$ and $\mathcal{Q}_q$ for $S^+V\mathcal{K}_q$). By induction it follows that $\mathcal{F}_*$ and $\mathcal{F}'_*$ are isomorphic, and therefore any two minimal resolutions are isomorphic.

The fact that $\epsilon' : \mathcal{F}'_0 :\to \mathcal{M}$ is an epimorphism implies that there exists a graded vector subspace $\mathcal{Q}' \subseteq \mathcal{F}'_0$ such that ϵ' restricts to give an isomorphism $\mathcal{Q}' \to \mathcal{Q}$. This induces a morphism $\alpha : \mathcal{F}_0 = SV \otimes_F \mathcal{Q} \to \mathcal{F}'_0$ which commutes with the augmentation morphisms. We claim that it is an isomorphism.

Observe that $\mathcal{Q}'$ does not intersect $S^+V\mathcal{F}'_0$. Now, let $x \in \mathcal{F}'_0$. Write $\epsilon'(x) = y + z$ for $y \in \mathcal{Q}$ and $z \in S^+V\mathcal{M}$. Let y' be the element of $\mathcal{Q}'$ such that $\epsilon'(y') = y$, and choose $z' \in S^+V\mathcal{F}'_0$ such that $\epsilon'(z') = z$. Then $\epsilon'(x - y' - z') = 0$, and therefore, by minimality of $\mathcal{F}'_*$, we have that $x - y' - z'$ belongs to $S^+V\mathcal{F}'_0$. This shows that $\mathcal{Q}'$ complements $S^+V\mathcal{F}'_0$.

Note that the multiplication map $SV \otimes_F \mathcal{Q}' \to \mathcal{F}'_0$ is an epimorphism. Moreover, the fact that $\mathcal{Q}'$ complements $S^+V\mathcal{F}'_0$ implies that the graded pieces of $\mathcal{Q}'$ have the same dimension as any set of generators of $\mathcal{F}'_0$. Therefore, checking the dimension of each graded piece we see that $SV \otimes_F \mathcal{Q}' \to \mathcal{F}'_0$ has to be an isomorphism. From this, it follows that the map $\alpha : \mathcal{F}_0 = SV \otimes_F \mathcal{Q} \to \mathcal{F}'_0$ is an isomorphism, as we wanted to show. □

1.2 Consider the field F as a module concentrated in degree 0 (so that the action of V is trivial). The polynomial Poincaré lemma states that there is a minimal resolution

$$\ldots \to SV \otimes_F \bigwedge^2 V \to SV \otimes_F V \to SV \to F \to 0 \tag{3}$$

where we consider $\bigwedge^q V$ as a graded vector space concentrated in degree q. The maps

$$SV \otimes_F \bigwedge^{q+1} V \to SV \otimes_F \bigwedge^q V \tag{4}$$

are induced by the vector space maps

$$\bigwedge^{q+1} V \to V \otimes_F \bigwedge^q V \tag{5}$$

which are given by the inclusion $\bigwedge^{q+1} V \subseteq V \otimes_F \bigwedge^q V$ inside $V^{\otimes(q+1)}$.

Concretely, if $v_1, \ldots, v_n$ is a basis for V, the map (5) is given by

$$v_{i_1} \wedge \ldots \wedge v_{i_{q+1}} \mapsto \sum_{j=1}^{q+1} (-1)^j v_{i_j} \otimes (v_{i_1} \wedge \ldots \widehat{v_{i_j}} \ldots \wedge v_{i_n}) \tag{6}$$

Tensoring (3) with a finite dimensional vector space W gives a minimal resolution of W considered as a SV module concentrated in degree 0.

1.3 For $\mathcal{M}, \mathcal{N}$ modules in $\mathrm{QFGMod}^{\geq 0}_{\mathrm{SV}}$, the tensor product $\mathcal{M} \otimes_{SV} N$ has the structure of a non-negatively graded QFG SV module. To see this, consider the vector space $\mathcal{M} \otimes_F \mathcal{N}$. It has a natural grading, defined by $(\mathcal{M} \otimes_F \mathcal{N})^k = \bigoplus_j \mathcal{M}^j \otimes_F \mathcal{N}^{k-j}$, where we identify $\mathcal{M} \otimes_F \mathcal{N}$ with $\bigoplus_{i,j} \mathcal{M}^i \otimes_F \mathcal{N}^j$. The canonical vector space map $\mathcal{M} \otimes_F \mathcal{N} \to \mathcal{M} \otimes_{SV} \mathcal{N}$ is an epimorphism. Its kernel is the homogeneous subspace of $\mathcal{M} \otimes_F \mathcal{N}$ spanned by the elements of the form $vx \otimes y - x \otimes vy$, where $v \in V$, and $x \in \mathcal{M}$, $y \in \mathcal{N}$ are homogeneous. Therefore, there is an induced grading on $\mathcal{M} \otimes_{SV} \mathcal{N}$, and it is easily seen that it is compatible with the SV module structure. This defines a bifunctor

$$\otimes_{SV} : \mathrm{QFGMod}^{\geq 0}_{\mathrm{SV}} \times \mathrm{QFGMod}^{\geq 0}_{\mathrm{SV}} \to \mathrm{QFGMod}^{\geq 0}_{\mathrm{SV}} \tag{7}$$

which satisfies the usual properties.

Fix $\mathcal{N} \in \mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$. We define $\mathrm{Tor}^q(\cdot, \mathcal{N})$ as the q-th left derived functor of $\cdot \otimes_{SV} \mathcal{N}$. If we forget about the grading, this coincides with the usual Tor of modules over SV. We have $\mathrm{Tor}_q(\mathcal{M}, \mathcal{N}) = \mathrm{Tor}_q(\mathcal{N}, \mathcal{M})$ with the usual proof.

1.4 Let $\mathcal{M} \in \mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$. The *Koszul homology* of $\mathcal{M}$ in degree q is defined as $H_q(\mathcal{M}) = \mathrm{Tor}_q(F, \mathcal{M})$, where F is seen as a trivial SV module as before. In particular, observe that we have $H_0(\mathcal{M}) = \mathcal{M}/S^+V\mathcal{M}$. We may compute the spaces $H_q(\mathcal{M})$ using the above resolution of F. That is, the Koszul homology of $\mathcal{M}$ is the homology of the *Koszul complex*

$$0 \to \mathcal{M} \otimes_F \bigwedge^n V \to \ldots \to \mathcal{M} \otimes_F \bigwedge^2 V \to \mathcal{M} \otimes_F V \to \mathcal{M} \to 0 \tag{8}$$

From this we have that $H_q(M)$ vanishes for $q > n$.

Alternatively, as $H_q(\mathcal{M})$ is also $\mathrm{Tor}_q(\mathcal{M}, F)$, we may compute these spaces using a minimal resolution $\mathcal{F}_*$ of $\mathcal{M}$. That is, the Koszul homology is the homology of the complex

$$\ldots \to \mathcal{F}_2 \otimes_{SV} F \to \mathcal{F}_1 \otimes_{SV} F \to \mathcal{F}_0 \otimes_{SV} F \to 0 \tag{9}$$

Observe that $\mathcal{F}_q \otimes_{SV} F = \mathcal{F}_q/S^+V\mathcal{F}_q$. The fact that $\mathcal{F}_*$ is minimal implies that the differentials on (9) are all zero, and so $H_q(\mathcal{M}) = \mathcal{F}_q/S^+V\mathcal{F}_q$ for any minimal resolution $\mathcal{F}_*$. In particular, we have that $\mathcal{F}_q = 0$ for $q > n$, for any minimal resolution $\mathcal{F}_*$. Moreover, if $H_q(\mathcal{M}) = 0$ for some q, then the resolution $\mathcal{F}_*$ has length less than q, and so $H_{q'}(\mathcal{M}) = 0$ for all $q' \geq q$.

The action of V on $\mathcal{F}_q/S^+V\mathcal{F}_q$ is trivial, and so $H_q(\mathcal{M})$ is simply a graded vector space. We denote by $H_q^k(\mathcal{M})$ the degree k part of $H_q(\mathcal{M})$. It is common in the literature to denote this space by $H_q^{k-q}(\mathcal{M})$, however we shall stick to our grading, which is the natural one once one agrees to work within $\mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$. Using a minimal resolution of $\mathcal{M}$, it is easily seen that $H_q^k(\mathcal{M}) = 0$ for $k < q$. Moreover, we have the following

Proposition 1.4.1. *Let $\mathcal{M}$ be a finitely generated non-negatively graded SV module, and $q > 0$. There exists k_0 such that $H_q^k(\mathcal{M}) = 0$ for $k \geq k_0$.*

Proof. Since $\mathcal{M}$ is finitely generated and SV is a Noetherian ring, we have that the homology of (8) is finitely generated as a SV module. Since we know that the action of V on the homology is trivial, we have that $H_q(\mathcal{M})$ is a finite dimensional vector space, and so it vanishes above a certain degree. □

In fact, if $\mathcal{M}$ is the quotient of a free module $SV \otimes W^*$, where W is a finite dimensional vector space concentrated in degree 0, then the integer k_0 in the above proposition may be taken to only depend on $\dim W$ and the Hilbert function $P_{\mathcal{M}}(k) = \dim \mathcal{M}^k$ of $\mathcal{M}$. See [1] for a proof.

2 Spencer Cohomology

2.1 Let V be an n-dimensional vector space over a field F of characteristic zero. Consider the dual coalgebra SV^* to the symmetric algebra SV. We say that a graded SV^* comodule $\mathcal{A}$ is *quasi-finitely generated* if the graded pieces $\mathcal{A}^k$ are finite dimensional vector spaces. Let $\mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$ be the category of QFG non-negatively graded SV^* comodules and degree preserving morphisms. Observe that $\mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$ and $\mathrm{QFGMod}^{\geq 0}_{\mathrm{SV}}$ are dual categories, the anti-equivalence being given by taking dual as graded vector spaces. That is, given $\mathcal{A} \in \mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$, its associated graded module is $\mathcal{M} = \oplus_k \hom(\mathcal{A}^k, F)$, and in the same way, if $\mathcal{M} \in \mathrm{QFGMod}^{\geq 0}_{\mathrm{SV}}$, its associated graded comodule is $\mathcal{A} = \oplus_k \hom(\mathcal{M}^k, F)$. Therefore, the abelian category $\mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$ has enough injectives. In fact, every object has a canonical isomorphism class of resolutions, which may be called *minimal*.

Observe that if $\mathcal{A}$ is a QFG graded vector space, graded SV^* comodule structures $\mathcal{A} \to \mathcal{A} \otimes_F SV^*$ are in correspondence with SV module structures $\mathcal{A} \otimes_F SV \to \mathcal{A}$ such that the action of V is homogeneous of degree -1. This is dual to the action of V on $\mathcal{M}$ by multiplication. We say that V acts on $\mathcal{A}$ by *contraction*. In the case that $\mathcal{A} = SV^*$, this is the usual contraction $V \otimes_F SV^* \to SV^*$.

2.2 Let $A, A' \in \mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$, and let δ, δ' be their respective comultiplications. The *cotensor product* $A \boxtimes A'$ is defined as the kernel of the map

$$\mathcal{A} \otimes_F \mathcal{A}' \to \mathcal{A} \otimes_F SV^* \otimes_F \mathcal{A}' \tag{10}$$

sending $x \otimes y$ to $\Delta(x) \otimes y - x \otimes \Delta(y)$. This is a graded SV^* comodule, dual to the tensor product $\mathcal{M} \otimes_{SV} \mathcal{M}'$ of the associated graded modules. Observe that $\mathcal{A} \boxtimes F = \ker \Delta$, where F is consider as a graded comodule concentrated in degree zero.

Fix $\mathcal{A}' \in \mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$. The functor $\cdot \boxtimes \mathcal{A}' : \mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}} \to \mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$ is left exact. The *Cotor functors* are the right derived functors of $\cdot \boxtimes \mathcal{A}'$, and are denoted $\mathrm{Cotor}^q(\cdot, A')$. Of course, $\mathrm{Cotor}^q(A, A')$ is just the dual comodule to $\mathrm{Tor}^q(\mathcal{M}, \mathcal{M}')$.

2.3 Let $\mathcal{A} \in \mathrm{QFGCoMod}^{\geq 0}_{\mathrm{SV}}$. The *Spencer cohomology* of $\mathcal{A}$ in degree q is defined as $H^q(\mathcal{A}) = \mathrm{Cotor}^q(\mathcal{A}, F)$. This is dual to the Koszul homology of the associated graded module. In particular, the comultiplication $H^q(\mathcal{A}) \to H^q(\mathcal{A}) \otimes_F SV^*$ is trivial, and one has $H^q(\mathcal{A}) = 0$ for $q > n$. We denote by $H^{q,k}(\mathcal{A})$ the k-th graded piece of $H^q(\mathcal{A})$. As before, we have that $H^{q,k}(\mathcal{A}) = 0$ for $k < q$, and there exists k_0 such that $H^{q,k}(\mathcal{A}) = 0$ for $k \geq k_0$.

We may compute these spaces using the dual of the Koszul complex. This gives a complex

$$0 \to \mathcal{A} \to \mathcal{A} \otimes_F V^* \to \mathcal{A} \otimes_F \Lambda^2 V^* \to \ldots \to \mathcal{A} \otimes_F \Lambda^n V^* \to 0 \tag{11}$$

where $\Lambda^q V^*$ is consider as a graded comodule concentrated in degree q. The first map $\mathcal{A} \to \mathcal{A} \otimes_F V^*$ is the comultiplication. In general, the map $\mathcal{A} \otimes_F \Lambda^q V^* \to \mathcal{A} \otimes_F \Lambda^{q+} V^*$ is given by the composition

$$\mathcal{A} \otimes_F \Lambda^q V^* \to \mathcal{A} \otimes_F V^* \otimes_F \Lambda^q V^* \to \mathcal{A} \otimes_F \Lambda^{q+1} V^* \tag{12}$$

where the first map is induced by comultiplication, and the second map is induced by the wedge product. Observe that the complex (11) is a right ΛV^* module, and the differential is ΛV^* linear.

Concretely, if $v_1, \ldots, v_n$ is a basis for V and $v^1, \ldots, v^n$ is its dual basis, the differentials are given by

$$x \otimes (v^{i_1} \wedge \ldots \wedge v^{i_q}) \mapsto \sum_{j=1}^{n} v_j \lrcorner x \otimes (v^j \wedge v_{i_1} \wedge \ldots \wedge v_{i_q}) \tag{13}$$

where $v_j \lrcorner x$ denotes contraction by v_j.

2.4 Let V, W be finite dimensional vector spaces over F. Let $\mathcal{A} \subseteq S^{\leq k} V^* \otimes_F W$ be a graded subcomodule. That is, $\mathcal{A}$ is a graded subspace of the space of polynomials of order at most k, closed under contraction. We may extend $\mathcal{A}$ to higher degrees, by setting

$$\mathcal{A}^{k+l} = (S^l V^* \otimes_F W^k) \cap (S^{k+l} V^* \otimes_F W) \tag{14}$$

for $l \geq 0$. That is, $\mathcal{A}^{k+l}$ consists of those polynomials of order $k+l$ such that all their derivatives of order l belong to $\mathcal{A}^k$. The space $\mathcal{A}$ is now a graded subcomodule of $SV^* \otimes_F W$. Its associated graded module $\mathcal{M}$ is a quotient of $SV \otimes_F W^*$. Let $\mathcal{B}$ be the kernel of the quotient map. The way that $\mathcal{A}$ was constructed implies that $\mathcal{B}$ is the span of $\mathcal{B}^{\leq k}$.

From the long exact sequence induced by

$$0 \to \mathcal{B} \to SV \otimes_F W^* \to \mathcal{M} \to 0 \tag{15}$$

we get $H_q(\mathcal{B}) = H_{q+1}(\mathcal{M})$ for $q \geq 0$. In particular, we have that $H^1(\mathcal{A})$ vanishes in degrees greater than k.

A particular case of interest is when $\mathcal{A}$ is obtained from a comodule of the form $(S^{<k} V^* \otimes_F W) \oplus A$ with $A \subseteq S^k V^* \otimes_F W$ a subspace. Here we use the notation $H^q(A) = H^q(\mathcal{A})$. The space A is called a *k-th order tableau*. The $(k+l)$-th order tableau $A^{(l)} = \mathcal{A}^{k+l}$ is called the *l-th prolongation* of A.

Example 2.4.1. Let V be an n-dimensional vector space over F and set $W = V^*$. Consider the subspace $A \subseteq V^* \otimes V^*$ of symmetric bilinear forms. Then $\mathcal{A} = S^{[1]} V^*$ where $[d]$ denotes a shift in degree, so that $(S^{[1]} V^*)^j = S^{j+1} V^*$ (and we drop the space S^0V^*). By the polynomial Poincaré lemma (i.e., the vanishing of the Spencer cohomology for SV^*), we have $H^{q,j}(A) = 0$ for $j > q \geq 0$.

Now, for $j = q$ we have

$$H^{q,q}(A) = \mathcal{A}^0 \otimes_F \Lambda^q V^*/\operatorname{im}\delta^{q-1} = V^* \otimes_F \Lambda^q V^*/\operatorname{im}\delta^{q-1} \tag{16}$$

where $\delta^{q-1} : S^2V^* \otimes_F \Lambda^{q-1}V^* \to V^* \otimes_F \Lambda^q V^*$ is the Spencer coboundary. Using the polynomial Poincaré lemma again, we get $H^{q,q}(A) = \Lambda^{q+1}V^*$ for $q \geq 0$.

EXAMPLE 2.4.2. Let V be an n-dimensional vector space over F and set $W = V^*$. Consider the subspace $A \subseteq V^* \otimes V^*$ of anti-symmetric bilinear forms. The space $\mathcal{A}^2$ is the space of trilinear forms on V which are symmetric in the first two entries and anti-symmetric in the last two entries. It is easily seen that these forms necessarily vanish, so we have $\mathcal{A}^2 = 0$. Therefore, $H^{q,j}(A) = 0$ for $j \geq q + 2$.

When $j = q$ we have

$$H^{q,q}(A) = V^* \otimes_F \Lambda^q V^*/\operatorname{im}\delta^{q-1} \tag{17}$$

where $\delta^{q-1} : A \otimes_F \Lambda^{q-1}V^* \to V^* \otimes_F \Lambda^q V^*$ is the Spencer coboundary. From this, we get $H^{1,1}(A) = S^2V^*$. Now, let $x, y, z \in V^*$. We have

$$x \otimes y \wedge z = \delta^1(x \wedge y \otimes z - y \wedge z \otimes x + z \wedge x \otimes y)/2 \tag{18}$$

and therefore $\delta^1 : A \otimes_F V^* \to V^* \otimes_F \Lambda^2 V^*$ is surjective. The fact that δ is right ΛV^*-linear implies that δ^{q-1} is surjective for $q \geq 2$. Therefore, $H^{q,q}(A) = 0$ for $q \geq 2$.

The only spaces left are those of the form $H^{q,q+1}(A)$. This is the kernel of $\delta^q : A \otimes_F \Lambda^q V^* \to V^* \otimes_F \Lambda^{q+1} V^*$. Since $\mathcal{A}$ is obtained from the comodule $V^* \oplus A \subseteq S^1V^* \otimes W$, we have $H^{1,2}(A) = 0$. When $q \geq 2$, we shall only mention that the space $H^{q,q+1}(A)$ may be non-zero. One may try to decompose this space into irreducible $GL(V)$ modules, however we shall not discuss this here.

EXAMPLE 2.4.3. Let V, W be two finite dimensional vector spaces over the real numbers, equipped with complex structures. Consider the subspace $A \subseteq V^* \otimes_{\mathbb{R}} W$ of complex linear transformations. Then $\mathcal{A} = S_{\mathbb{C}} V^\vee \otimes_{\mathbb{C}} W$, where $V^\vee$ denotes the dual as a complex vector space, and $S_{\mathbb{C}}$ denotes the symmetric algebra as a complex vector space. Of course, the complex polynomial Poincaré lemma tells us that the cohomology of $\mathcal{A}$ as a $S_{\mathbb{C}} V^\vee$ comodule vanishes, however we are interested in its cohomology as a SV^* comodule. This is the cohomology of the complex

$$0 \to \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{R}} V^* \to \mathcal{A} \otimes_{\mathbb{R}} \Lambda^2 V^* \ldots \to \mathcal{A} \otimes_{\mathbb{R}} \Lambda^{2n} V^* \to 0 \tag{19}$$

Observe that $\mathcal{A} \otimes_{\mathbb{R}} \Lambda^q V^* = \mathcal{A} \otimes_{\mathbb{C}} \Lambda^q_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C})$, where $\Lambda_{\mathbb{C}}$ denotes the exterior powers as a complex vector space. Therefore, (19) becomes

$$0 \to \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{C}} (V^* \otimes_{\mathbb{R}} \mathbb{C}) \to \mathcal{A} \otimes_{\mathbb{C}} \Lambda^2_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C}) \ldots \to \mathcal{A} \otimes_{\mathbb{C}} \Lambda^{2n}_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C}) \to 0 \tag{20}$$

Denote by $\overline{V}^\vee$ the space of complex anti-liner functionals $V \to \mathbb{C}$. The decomposition of $V^* \otimes_{\mathbb{R}} \mathbb{C}$ as $V^\vee \oplus \overline{V}^\vee$ induces a bigrading on its exterior algebra. We denote by $\Lambda^{r,s}_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C})$ the space generated by wedges of r linear and s anti-linear functionals. This induces a bigrading on (20). Consider the spectral sequence corresponding to the filtration induced by $F^s(\Lambda_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C})) = \bigoplus_{r\geq 0} \Lambda^{r,s}_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C})$. The zeroth page consists of the spaces

$$(\mathcal{A} \otimes_{\mathbb{C}} \Lambda^{r,s}_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C}))/(\mathcal{A} \otimes_{\mathbb{C}} \Lambda^{r,s+1}_{\mathbb{C}}(V^* \otimes_{\mathbb{R}} \mathbb{C})) = \mathcal{A} \otimes_{\mathbb{C}} \Lambda^r_{\mathbb{C}} V^\vee \otimes_{\mathbb{C}} \Lambda^s_{\mathbb{C}} \overline{V}^\vee \tag{21}$$

These spaces form complexes of the form

$$0 \to \mathcal{A} \otimes_{\mathbb{C}} \Lambda^s_{\mathbb{C}} \overline{V}^\vee \to \mathcal{A} \otimes_{\mathbb{C}} V^\vee \otimes_{\mathbb{C}} \Lambda^s_{\mathbb{C}} \overline{V}^\vee \to \ldots \to \mathcal{A} \otimes_{\mathbb{C}} \Lambda^n_{\mathbb{C}} V^\vee \otimes_{\mathbb{C}} \Lambda^s_{\mathbb{C}} \overline{V}^\vee \to 0 \tag{22}$$

This is the tensor product with $\Lambda^s_{\mathbb{C}} \overline{V}$ of the Spencer complex of $\mathcal{A}$ as a $S_{\mathbb{C}} V^\vee$ comodule. By the complex polynomial Poincaré lemma, the first page has $E_1^{r,s} = 0$ for $r > 0$, and $E_1^{0,s} = W \otimes_{\mathbb{C}} \Lambda^s_{\mathbb{C}} \overline{V}^\vee$, which is a complex vector space concentrated in degree s. This implies that the differential $E_1^{0,s} \to E_1^{0,s+1}$ vanishes, and so the sequence already converges in the first page. Since this sequence computes the cohomology of (20), we have that $H^{q,j}(A) = 0$ if $j > q \geq 0$, and $H^{q,q}(A) = W \otimes_{\mathbb{C}} \Lambda^q_{\mathbb{C}} \overline{V}^\vee$ for $q \geq 0$.

3 Involution

3.1 Let F be a field of characteristic zero, and V be an n-dimensional vector space over F. Let $\mathcal{M}$ be a QFG non-negatively graded SV module. An element $v \in V$ is said to be *regular* for $\mathcal{M}$ if multiplication by v is a monomorphism. A sequence $v_1, \ldots, v_{n+1-q}$ of elements in V is said to be a *regular sequence* for $\mathcal{M}$ if v_j is regular for $\mathcal{M}/(v_1, \ldots, v_{j-1})\mathcal{M}$ for all $1 \leq j \leq n+1-q$.

The existence of regular sequence is related to the vanishing of the Koszul homology of $\mathcal{M}$, as follows

Proposition 3.1.1. *Let $\mathcal{M}$ be a QFG non-negatively graded SV module, and $0 \leq q \leq n$. If there exists a regular sequence of length $n+1-q$ for $\mathcal{M}$, then $H_q(\mathcal{M})$ vanishes. Conversely, if $\mathcal{M}$ is finitely generated and $H_q(\mathcal{M})$ vanishes, then a generic sequence $v_1, \ldots, v_{n+1-q}$ is regular for $\mathcal{M}$.*

Proof. Consider first the case $q = n$. Suppose that there is a v which is regular for $\mathcal{M}$, and complete it to a basis for V. From the expression of the Koszul boundary in this basis it is easily seen that $\delta_n : \mathcal{M} \otimes_F \Lambda^n V \to \mathcal{M} \otimes_F \Lambda^{n-1} V$ is a monomorphism, and so $H_n(\mathcal{M}) = 0$.

Conversely, suppose that $H_n(\mathcal{M})$ vanishes. Let $m \in \mathcal{M} - \{0\}$, and $v_1, \ldots, v_n$ be a basis for V. The fact that $\delta_n(m \otimes v_1 \wedge \ldots \wedge v_n) \neq 0$ implies that there is a index i such that $v_i m \neq 0$. Therefore, SV is not an associated prime for $\mathcal{M}$. Since we assume that

$\mathcal{M}$ is finitely generated, there are only a finite number of associated primes $\mathcal{P}_1, \ldots, \mathcal{P}_r$ for $\mathcal{M}$. The non-regular elements for $\mathcal{M}$ are then the elements of $\cup_i \mathcal{P}_i^1 \subseteq V$, which is a finite union of proper subspaces of V.

Now, suppose that $v \in \mathcal{M}$ is a regular element, and consider the short exact sequence

$$0 \to \mathcal{M}^{[-1]} \xrightarrow{\mu_v} \mathcal{M} \to \mathcal{M}/v\mathcal{M} \to 0 \tag{23}$$

where μ_v is multiplication by v, and $[-1]$ denotes a shift in the grading so that μ_v preserves degree. The action of V on the Koszul homology is always trivial, so the maps induced by μ_v on the homology are zero. Therefore, from (23) we get short exact sequences

$$0 \to H_i(\mathcal{M}) \to H_i(\mathcal{M}/v\mathcal{M}) \to H_{i-1}(\mathcal{M})^{[-1]} \to 0 \tag{24}$$

from which the proposition follows, by induction. □

We say that a module $\mathcal{M}$ in $\mathrm{QFGMod}_{\mathrm{SV}}^{\geq 0}$ is *l-involutive* if $H_q^j(\mathcal{M}) = 0$ for all $q \geq 0$ and $j \geq q + l$. By proposition 1.4.1, any finitely generated module is l-involutive for some l. An element $v \in V$ is said to be l-regular for $\mathcal{M}$ if multiplication by v is a monomorphism in degrees greater or equal to l. A sequence $v_1, \ldots, v_{n+1-q}$ of elements in V is said to be a *l-regular sequence* for $\mathcal{M}$ if v_j is l-regular for $\mathcal{M}/(v_1, \ldots, v_{j-1})\mathcal{M}$ for all $1 \leq j \leq n + 1 - q$. The following proposition follows from the same kind of arguments as proposition 3.1.1

Proposition 3.1.2. *Let $\mathcal{M}$ be a QFG non-negatively graded SV-module, and $0 \leq q_0 \leq n$. If there exists a l-regular sequence of length $n + 1 - q_0$ for $\mathcal{M}$, then $H_q^j(\mathcal{M}) = 0$ for all $q \geq q_0$ and $j \geq q + l$. Conversely, if $\mathcal{M}$ is finitely generated and $H_q^j(\mathcal{M}) = 0$ for all $q \geq q_0$ and $j \geq q + l$, a generic sequence $v_1, \ldots, v_{n+1-q}$ is regular for $\mathcal{M}$.*

In particular, it follows that l-involutivity is equivalent to the existence of l-regular sequences of length n.

3.2 Let $\mathcal{A}$ be a QFG non-negatively graded SV^* comodule. A sequence of elements $v_1, \ldots, v_{n+1-q}$ in V is said to be *l-regular* for $\mathcal{A}$ if it is l-regular for its associated graded module $\mathcal{M}$. Denote by $\iota_v : \mathcal{A} \to \mathcal{A}$ the contraction by $v \in V$, and let

$$\mathcal{A}_j = \ker \iota_{v_1} \cap \ldots \cap \ker \iota_{v_j} \tag{25}$$

Then $v_1, \ldots, v_{n+1-q}$ is l-regular if and only if $\iota_{v_{j+1}} : \mathcal{A}_j^{\geq l+1} \to \mathcal{A}_j^{\geq l}$ is surjective for all $0 \leq j \leq n - q$. We say that $\mathcal{A}$ is *l-involutive* if $H^{q,j}(\mathcal{A}) = 0$ for all $q \geq 0$ and $j \geq q + l$. Of course, the existence of l-regular sequences is related to l-involutivity, via the dual to proposition 3.1.2.

3.3 Let V, W be two finite dimensional vector spaces over F, with $\dim V = n$. Let $A \subseteq S^k V^* \otimes W$ be a k-th order tableau, and $\mathcal{A} \subseteq SV^* \otimes W$ be the associated SV^* comodule. We say that A is *involutive* if $\mathcal{A}$ is k-involutive. By proposition 1.4.1, we have that $A^{(l)}$ is involutive for l large enough.

A *quasi-regular sequence* for A is a k-regular sequence for $\mathcal{A}$. By proposition 3.1.2, the involutivity of A is equivalent to the existence of a quasi-regular sequence of length n. If $v_1, \ldots, v_n$ is a sequence of elements in V, let A_j be the subspace of A consisting of those polynomials P such that $v_i \lrcorner P = 0$ for all $i \leq j$. Observe that A_j may be thought of as a k-th order tableau in two different ways, corresponding to the coalgebras SV^* and $S(V/(v_1))^*$. Both notions of prolongation coincide, and we have $(A_j)^{(l)} = (A^{(l)})_j$.

PROPOSITION 3.3.1. *A sequence $v_1, \ldots, v_n$ is quasi-regular for A if and only if the contraction $\iota_{v_{j+1}} : A_j^{(1)} \to A_j$ is surjective for all $0 \leq j \leq n-1$.*

Proof. Of course, quasi-regularity implies that all those contractions are surjective, so we only need to show the converse. Assume that it holds when $\dim V = n-1$, so we only need to prove that $\iota_{v_1} : \mathcal{A} \to \mathcal{A}$ is surjective. By induction on k, we only need to show that $\iota_{v_1} : A^{(2)} \to A^{(1)}$ is surjective.

Let $Q \in A^{(1)}$ and consider the 1-form $\delta^0(Q) \in A \otimes V^*$, where δ^0 denotes the zeroth Spencer coboundary. By hypothesis, there exists $T \in A^{(1)} \otimes V^*$ such that $\iota_{v_1} T = \delta^0(Q)$. We have that $\iota_{v_1} \delta^1 T = \delta^1 \delta^0 Q = 0$, and so $\delta^1 T$ belongs to $A_1 \otimes \Lambda^2 V^*$. We claim that we could have chosen T so that $\delta^1 T = 0$.

To see that, let $V_1^* = (V/(v_1))^*$, and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1^{(1)} \otimes V_1^* & \longrightarrow & A_1^{(1)} \otimes V^* & \longrightarrow & A_1^{(1)} \otimes (\mathbb{R}v_1)^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 \otimes \Lambda^2 V_1^* & \longrightarrow & A_1 \otimes \Lambda^2 V^* & \longrightarrow & A_1 \otimes (\mathbb{R}v_1)^* \otimes V_1^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W \otimes \Lambda^3 V_1^* & \longrightarrow & W \otimes \Lambda^3 V^* & \longrightarrow & W \otimes (\mathbb{R}v_1)^* \otimes \Lambda^2 V_1^* & \longrightarrow & 0
\end{array} \tag{26}$$

The left column computes the cohomology $H^{2,k+2}(\mathcal{A}_1)$ as a SV_1^* comodule, which vanishes since A_1 is involutive by induction. The right column computes $H^{1,k+1}(\mathcal{A}_1) \otimes (\mathbb{R}v_1)^* = 0$. From this, we see that the middle column is exact. Since $\delta^1 T \in A_1 \otimes \Lambda^2 V^*$ is δ^2 closed, it follows that we may change T so that $\delta^1 T$ vanishes, as we claimed.

By the polynomial Poincaré lemma, there exists $P \in A^{(2)}$ such that $T = \delta^0 P$. From this, we have

$$\delta^0 \iota_{v_1} P = \iota_{v_1} \delta^0 P = \iota_{v_1} T = \delta^0 Q \tag{27}$$

and so $\iota_{v_1} P = Q$. Since Q was arbitrary, it follows that $\iota_{v_1} : A^{(2)} \to A^{(1)}$ is surjective, as we wanted. □

Corollary 3.3.2 (Cartan's test). *Let A be a k-th order tableau and $v_1, \ldots, v_n$ be a basis for V. We have*

$$\dim A^{(1)} \leq \dim A + \dim A_1 + \ldots + \dim A_{n-1} \tag{28}$$

with equality if and only if $v_1, \ldots, v_n$ is a quasi-regular sequence for A.

Proof. Consider for each $0 \leq j \leq n-1$ the following exact sequence

$$0 \to A^{(1)}_{j+1} \to A^{(1)}_j \xrightarrow{\iota_{v_{j+1}}} A_j \tag{29}$$

Counting dimensions, we have

$$\dim A^{(1)}_j \leq \dim A^{(1)}_{j+1} + \dim A_j \tag{30}$$

for all $0 \leq j \leq n-1$, which proves the inequality in the statement. By proposition 3.3.1, $v_1, \ldots, v_n$ is quasi-regular for A if and only if all the sequences (29) are exact. This happens if and only we have equality in (30) for all j, which is equivalent to the equality in the statement. □

3.4 Let $X \subseteq V$ be a j-dimensional subspace. Consider the subspace A_X of A consisting of those polynomials $P \in A$ such that $v \lrcorner P = 0$ for all $v \in X$. The dimension of A_X is upper semicontinuous with respect to X. Moreover, the minimum dimension is attained for X in an open dense subset of the Grassmanian $G_j(V)$. We say that X is *generic* (with respect to A) if the dimension of A_X is minimal. An ordered basis $v_1, \ldots, v_n$ for V is said to be *generic* if the subspaces $(v_1, \ldots, v_j)$ are generic for all $1 \leq j \leq n$. By Cartan's test, when A is involutive, a basis is generic if and only if it is a quasi-regular sequence.

The *Cartan characters* of A are the integers $s_1, \ldots, s_n$ such that for all $0 \leq j \leq n$ we have

$$\dim A_X = s_{j+1} + \ldots + s_n \tag{31}$$

for a generic $X \in G_j(V)$. Alternatively, they may be defined as $s_j = \dim A_{j-1} - \dim A_j$ for any generic basis $v_1, \ldots, v_n$. Cartan's test may be reformulated as follows

Proposition 3.4.1. *Let A be a k-th order tableau. We have*

$$\dim A^{(1)} \leq s_1 + 2s_2 + \ldots + ns_n \tag{32}$$

with equality if and only if A is involutive.

Moreover by proposition 3.1.1 we have that $H^q(A)$ vanishes in all degrees if and only if $s_1 = \ldots = s_{n+1-q} = \dim W$.

3.5 The following result will let us put first order tableaux in a normalized form in coordinates.

PROPOSITION 3.5.1. *Let A be a first order tableau, and $v_1, \dots, v_n$ be a generic basis for A. For each $1 \leq j \leq n$, let W_j be the image of the contraction $\iota_{v_j} : A_{j-1} \to W$. Then $W_{j+1} \subseteq W_j$ for all $1 \leq j \leq n-1$.*

Proof. Take $P_1, \dots P_{s_1}$ linearly independent elements of A such that $v_1 \lrcorner P_i$ form a basis for W_1. Suppose that W_2 is not contained inside W_1, so that there is $Q \in A_1$ such that $v_2 \lrcorner Q \notin W_1$. Let $\bar{v}_1 = v_1 + \varepsilon v_2$, for some $\varepsilon \in \mathbb{R}$. If ε is sufficiently small, we have that $\bar{v}_1 \lrcorner P_1, \bar{v}_1 \lrcorner P_2, \dots, \bar{v}_1 \lrcorner P_{s_1}, \bar{v}_1 \lrcorner Q$ are $s_1 + 1$ linearly independent elements of $\iota_{v_1}(A)$. This contradicts the fact that v_1 is generic, and so we must have $W_2 \subseteq W_1$. The case $j > 1$ follows by applying this result for the first order tableau A_{j-1}. □

COROLLARY 3.5.2. *The Cartan characters of any k-th order tableau satisfy*

$$s_1 \geq s_2 \geq \dots \geq s_n \tag{33}$$

Proof. The case $k = 1$ is a direct consequence of the above proposition, since $\dim W_j = s_j$. In general, any k-th order tableau $A \subseteq S^k V^* \otimes W$ may be interpreted as a first order tableau contained in $V^* \otimes (S^{k-1} V^* \otimes W)$, so the corollary also holds in this case. □

Let $A \subseteq V^* \otimes W$ be a first order tableau. Let $v_1, \dots, v_n$ be a generic basis, and $w_1, \dots, w_s$ be a basis for W such that W_j is spanned by $w_1, \dots, w_{s_j}$. Let v^i and w^a be the dual basis. Then, for each $1 \leq j \leq n$, the quotient $A_{j-1}/A_j \subseteq (\mathbb{R}v_j)^* \otimes W$ is spanned by the vectors $v^j \otimes w_a$ with $a \leq s_j$. Therefore, A has a basis of the form

$$v^j \otimes w_a + A^{jb}_{ai} v^i \otimes w_b \tag{34}$$

where we have $1 \leq j \leq n$, $a \leq s_j$, and $A^{jb}_{ai} = 0$ unless $i > j$ and $b > s_i$. Dually, the annihilator $B \subseteq V \otimes W^*$ of A has a basis of the form

$$v_j \otimes w^a + B^{ai}_{jb} v_i \otimes w^b \tag{35}$$

where we have $1 \leq j \leq n$, $a > s_j$, and $B^{ai}_{jb} = 0$ unless $i < j$ and $b \leq s_i$.

The tableau A is sometimes represented as a matrix, and this says that the matrix may be taken to have a specific block form. In the case that A is involutive there are extra relations that the coefficients A^{jb}_{ai} must satisfy. This leads to the Guillemin Normal form for involutive tableaux (see [1]). As a first step in that direction, we offer the following

PROPOSITION 3.5.3. *If A is involutive, we have that $A^{jb}_{ai} = 0$ when $b > s_j$.*

Proof. By induction, we may assume that it holds when $j >$, so we must prove that $A^{1b}_{ai} = 0$ when $b > s_1$. Let $1 \leq a \leq s_1$, and choose $P \in A^{(1)}$ such that

$$v_1 \lrcorner P = v_1 \otimes w^a + A^{1b}_{ai} v^i \otimes w_b \tag{36}$$

For each $i > 1$, we have that $v_i \lrcorner (v_1 \lrcorner P) \in W_1$. Therefore $A^{1b}_{ai} w_b \in W_1$, so we must have $A^{1b}_{ai} = 0$ for $b > s_1$, as we claimed. □

It may be seen that if A is a tableau such that those coefficients vanish, involutivity is equivalent to a quadratic condition on the remaining coefficients.

Chapter III

Differential Equations

In this chapter we present the basic theory of differential operators and equations using the jet formalism developed in chapter I.

Section 1 deals with differential operators. We begin by discussing the general case of k-th order differential operators sending n-dimensional submanifolds of a manifold M to n-dimensional submanifolds of a manifold M'. These may be prolonged to form higher order differential operators. It turns out that the prolongations respect the affine structure on the jet spaces. The associated vector bundle maps depend on the principal symbol of the operator, which is an object which describes its highest order behavior. Finally, these concepts are adapted to the case of differential operators acting on sections of a fibered manifold.

In section 2 we study k-th order differential equations on n-dimensional submanifolds of a manifold M, that is, subsets $R \subseteq J^k_n(M)$. Assuming smoothness, these may be prolonged to form higher order differential equations having the same solutions as the original one. We discuss how one may present a differential equation using a differential operator, and the relationship between the prolongations of the equation and the operator. We introduce the principal and sub-principal symbols of the equation. From these (under mild conditions) one may construct a bundle $\mathcal{A}_R$ of graded SU^* comodules over R. This comodule governs the behavior of the prolongations of the equation. Lastly, we observe how this theory adapts to the case of differential equations on sections of a fibered manifold.

In section 3 we introduce the concept of formal integrability of a differential equation. If R is nonempty, this guarantees the existence of formal solutions (in coordinates, formal series solving the equation and all its derivatives). In the analytic category, it may be seen that formal integrability implies the existence of local solutions, however this is false in the C^∞ case without extra assumptions. We construct the first obstruction to integrability, which is a section of the bundle $H^{2,k+1}(\mathcal{A}_R)$. Proceeding inductively, we will prove the theorem of Goldschmidt which asserts that if $H^{2,k+l+1}(\mathcal{A}_R)$ vanishes for all $l \geq 0$, then R is formally integrable. To finish this section, we give an alternative construction of the curvature in the case where R is a differential equation on sections of a fibered manifold given by a differential operator. This depends on the description of the Spencer cohomology of $\mathcal{A}_R$ via minimal resolutions.

Section 4 deals with the initial value problem: given a k-th order differential equation $R \subseteq J^k_n(M)$ and (generic) initial conditions along a $n-1$ dimensional submanifold of M, we want conditions that assure that it is possible to extend this to a solution of R. It is easily seen that the initial conditions have to satisfy a first order equation (corresponding in coordinates to the commutativity of derivatives). We shall see that if this equation is formally integrable then the only obstruction to the solvability of the formal initial value problem lies in the first order (i.e., whether or not any generic initial conditions along a 1-jet of an $(n-1)$-dimensional submanifold satisfying the equation may be extended to a $(k+1)$-jet of a solution to R). This is related to the Cartan-Kahler existence theorem in the theory of analytic exterior differential systems.

1 Differential Operators

1.1 Let M, M' be two differentiable manifolds, and fix $n \leq \dim M$. Let $k \geq 0$. A *k-order differential operator* taking n-dimensional submanifolds of M to n-dimensional submanifolds of M' is a smooth function $\varphi : V \to M'$ where $V \subseteq J^k_n(M)$ is an open subspace. Given an n-dimensional submanifold $i : N \to M$ such that $N^{(k)} \subseteq V$ and $\varphi|_{T_y N^{(k)}}$ is a monomorphism for all $y \in N^{(k)}$, the composition $\Delta_\varphi(i) = \varphi \circ i^{(k)}$ is an immersion from N to M'. We denote by $\Delta_\varphi(N)$ the manifold N when seen as an immersed submanifold of M'. For simplicity of exposition, we assume that φ is globally defined on $J^k_n(M)$, and that it is a monomorphism when restricted to any integral element of the contact distribution on $J^k_n(M)$, so that $\Delta_\varphi(N)$ is defined for all N.

Strictly speaking, one should define a differential operator to be an operator Δ sending submanifolds of M to submanifolds of M', for which there exists a function φ as above such that $\Delta = \Delta_\varphi$. The order of Δ is then defined as the least k for which on may take φ to be defined on $J^k_n(M)$. Observe that even if φ is a function on $J^k_n(M)$, the operator Δ_φ may have order less than k. Having said this, we shall stick to our original definition most of the time.

Let N be an n-dimensional submanifold of M. Let $l \geq 0$, and consider the submanifold $\Delta_\varphi(N)^{(l)}$ inside $J^l_n(M')$. For each $q \in N$, the l-jet $\Delta_\varphi(i)^{(l)}(q)$ only depends on the $(k+l)$-jet of N at x, so we get a well defined map

$$\varphi^{(l)} : J^{k+l}_n(M) \to J^l_n(M') \tag{1}$$

which is a $(k+l)$-th order operator with values in $J^l_n(M')$ called the *l-th prolongation* of φ. We set $\Delta^{(l)}_\varphi = \Delta_{\varphi^{(l)}}$. By definition, $\Delta^{(l)}_\varphi(N) = \Delta_\varphi(N)^{(l)}$. Observe that we have the following commutative diagrams

$$\begin{array}{ccc} J_n^{k+l+1}(M) & \xrightarrow{\varphi^{(l+1)}} & J_n^{l+1}(M') \\ \downarrow & & \downarrow \\ J_n^{k+l}(M) & \xrightarrow{\varphi^{(l)}} & J_n^{l}(M') \end{array} \tag{2}$$

For zeroth order operators (i.e., functions), this notion of prolongation coincides with the prolongation defined in chapter I. If φ is of order $k > 0$, there is some ambiguity when talking about $\varphi^{(l)}$: it could also be interpreted as being the l-th prolongation of φ as a function, in which case it should be a map $J_n^l(J_n^k(M)) \to J_n^l(M')$. The l-th prolongation of φ as a differential operator is the restriction of this map to $J_n^{k+l}(M)$ via the canonical embedding $J_n^{k+l}(M) \subseteq J_n^l(J_n^k(M))$. From now on, we shall use $\varphi^{(l)}$ for the l-th prolongation of φ as a differential operator.

Consider the iterated prolongation $\varphi^{(l)(m)} : J_n^{k+l+m}(M) \to J_n^m(J_n^l(M'))$. The image of this map belongs to $J_n^{l+m}(M')$, and its co-restriction equals $\varphi^{(l+m)}$. One could therefore define the prolongations of φ in an inductive way using that $\varphi^{(l+1)} = \varphi^{(l)(1)}$, where the first prolongation $\varphi^{(1)}$ of an operator may be obtained as the restriction to $J_n^{k+1}(M)$ of the map $J_n^1(J_n^k(M)) \to J_n^1(M')$ induced by φ. Observe that this construction only depends on the space $J_n^k(M)$ and the distributions $\mathcal{C}_n^k$ and $V\pi_{k,k-1}$; we do not need to use that $J_n^k(M)$ is a space of jets.

Using the concept of prolongation, we may define the composition of two differential operators $\varphi : J_n^k(M) \to M'$ and $\psi : J_n^l(M') \to M''$ as the $(k+l)$-th order operator $\psi \circ \varphi^{(l)} : J_n^{k+l}(M) \to M''$. We use the notation $\Delta_\psi \circ \Delta_\phi = \Delta_{\psi\circ\varphi^{(l)}}$. Observe that $\Delta_\psi \circ \Delta_\varphi(N) = \Delta_\psi(\Delta_\varphi(N))$. It may happen that $\psi \circ \varphi^{(l)}$ is actually defined on a lower order jet space, in this case the operator $\Delta_\psi \circ \Delta_\varphi$ would have order less than $k + l$.

Example 1.1.1. Consider the case $M' = J_n^k(M)$ and $\varphi = \mathrm{id}_{J_n^k(M)}$. This is the *k-th order universal differential operator*, and is denoted by id_k. Any k-th order operator is then obtained by composing the universal operator with a function. We have $\Delta_{\mathrm{id}_k}(N) = N^{(k)}$. The l-th prolongation $\mathrm{id}_k^{(l)}$ is given by the canonical embedding $J_n^{k+l}(M) \hookrightarrow J_n^l(J_n^k(M))$.

1.2 Let U,U' be the universal bundles on $J_n^1(M)$ and $J_n^1(M')$. Let $Q = TM/U$ and $Q' = TM'/U'$. As usual, we pullback bundles on $J_n^l(M')$ to $J_n^{k+l}(M)$ via $\varphi^{(l)}$. As in the case of zeroth order operators, $\varphi_*^{(l)}$ gives an isomorphism $U^{(k+l)} = U'^{(l)}$ over $J_n^{k+l+1}(M)$. Unless we are interested in its embedding inside the tangent space to a particular jet space, we shall identify all the universal bundles and simply speak of U.

From now on, we assume $k \geq 1$. The *principal symbol* of the differential operator φ is the map of bundles over $J_n^{k+1}(M)$

$$\sigma_\varphi : V\pi_{k,k-1} = S^kU^* \otimes Q \to Q' \tag{3}$$

induced by φ_*. Again, there is some ambiguity to this: if we forget that φ is a differential operator and treat it as a function, its symbol is a map $J^k_n(Q \to J^1_n(M)) \to Q'$. When we restrict it to $S^kU^* \otimes Q$ we get the map (3). We shall always use σ_φ for the symbol of φ as a differential operator.

The *l-th prolongation* of the symbol is the map

$$\sigma^{(l)}_\varphi = S^{k+l}U^* \otimes Q \to S^lU^* \otimes Q' \tag{4}$$

given by the restriction of

$$1_{S^lU^*} \otimes \sigma_\varphi : S^lU^* \otimes (S^kU^* \otimes Q) \to S^lU^* \otimes Q' \tag{5}$$

to $S^{k+l}U^* \otimes Q$. These maps determine the behavior of the prolongations of φ, as follows

Proposition 1.2.1. *The map $V\pi_{k+l,k+l-1} \to V\pi_{l,l-1}$ induced by $\varphi^{(l)}_*$ may be identified with $\sigma^{(l)}_\varphi$ for each $l \geq 1$. Moreover, if $l \geq 2$, the map $\varphi^{(l)}$ is an affine bundle morphism over $\varphi^{(l-1)}$, with associated vector bundle map $\sigma^{(l)}_\varphi$.*

The proof of this goes along the same lines as the proof of proposition I.1.6.2, and is left to the reader.

1.3 Let M be a manifold and $n \leq \dim M$. Let $\xi : E_\xi \to M$ and $\eta : E_\eta \to J^k_n(M)$ be two fibered manifolds. A *k-th order differential operator* acting on sections of ξ over n-dimensional submanifolds of M with values in η is a differential operator $\varphi : J^k_n(\xi) \to E_\eta$ such that $\eta \circ \varphi = \xi^{(k)}$. In other words, this is a morphism of fibered manifolds $J^k_n(\xi) \to E_\eta$ over $J^k_n(M)$. For each pair (N, s) of an n-dimensional submanifold N of M and a section s of $\xi|_N$, we may write $\Delta_\varphi(N, s)$ as $(N^{(k)}, \Delta_\varphi(s))$ for a certain section $\Delta_\varphi(s)$ of $\eta|_{N^{(k)}}$. An important particular case of this is when η is pullbacked from M: here the operator may be interpreted as sending sections of ξ over N to sections of η over N.

The l-th prolongation of φ is a map $\varphi^{(l)} : J^{k+l}_n(\xi) \to J^l_n(\eta)$ of fibered manifolds over $J^{k+l}_n(M)$. Let x^i, u^a be coordinates on M, and extend them to coordinate systems x^i, u^a, v^b and x^i, u^a_I, v'^c on E_ξ and E_η. The operator φ may be written as

$$\varphi(x^i, u^a_I, v^b_J) = (x^i, u^a_I, \varphi^c(x^i, u^a_I, v^b_J)) \tag{6}$$

Then the l-th prolongation is given by the formula

$$\varphi^{(l)}(x^i, u^a_I, v^b_J) = (x^i, u^a_I, D_{J'}\varphi^c(x^i, u^a_I, v^b_J)) \tag{7}$$

where $D_{J'}\varphi^c$ denotes the iterated derivative of φ^c with respect to $x^{J'_1}, \ldots, x^{J'_l}$, treating the variables u^a_I and v^b_J as functions.

In this case, the definition of the symbol should be adapted to reflect the fibered nature of the spaces. The map $\pi_{k,k-1} : J^k_n(\xi) \to J^{k-1}_n(\xi)$ should now be considered as a morphism of fibered manifolds over $J^k_n(M)$. The symbol of φ is defined to be the map of bundles over $J^k_n(\xi)$

$$\sigma_\varphi : S^k U^* \otimes V\xi \to V\eta \tag{8}$$

induced by $\varphi_* : V\pi_{k,k-1} \to V\eta$. The l-th prolongation of the symbol is the map

$$\sigma^{(l)}_\varphi : S^{k+l} U^* \otimes V\xi \to S^l U^* \otimes V\eta \tag{9}$$

given by the restriction of

$$1_{S^l U^*} \otimes \sigma_\varphi : S^l U^* \otimes (S^k U^* \otimes Q) \to S^l U^* \otimes Q \tag{10}$$

to $S^{k+l}U^* \otimes Q$. Proposition 1.2.1 is still valid in this context when interpreted correctly. Moreover, $\varphi^{(l)}$ is a morphism of affine bundles over $\varphi^{(l-1)}$ for $l = 1$ too.

1.4 Consider now the case when $\xi : E_\xi \to M$ and $\eta : E_\eta \to J^k_n(M)$ are vector bundles. A *k-th order linear differential operator* acting on sections of ξ over n-dimensional submanifolds with values in η is a morphism of vector bundles $\varphi : J^k_n(\xi) \to E_\eta$. In this case, the prolongations $\varphi^{(l)}$ are also linear. The symbol of φ may be identified with the restriction of φ to $\ker(\pi_{k,k-1}) = S^k U^* \otimes Q$. More generally, $\varphi^{(l)}$ restricts to a map

$$\ker(\pi_{k+l,k+l-1} : J^{k+l}_n(\xi) \to J^{k+l-1}_n(\xi)) \to \ker(\pi_{l,l-1} : J^l_n(\eta) \to J^{l-1}_n(\eta)) \tag{11}$$

which may be identified with the prolonged symbol $\sigma^{(l)}_\varphi$.

Let M, M' be two manifolds and $\varphi : J^k_n(M) \to M'$ be a differential operator. Its *linearization* is the morphism

$$\ell_\varphi : J^k_n(Q \to J^1_n(M)) = TJ^k_n(M)/U^{(k)} \to Q' \tag{12}$$

of vector bundles over $J^{k+1}_n(M)$, induced by φ_*. This is a k-th order linear differential operator from Q to Q'.

In the case that we have a (nonlinear) differential operator $\varphi : J^k_n(\xi) \to E_\eta$ between fibered manifolds, the linearization is defined as the morphism

$$\ell_\varphi : J^k_n(V\xi \to E_\xi) = V\xi^{(k)} \to V\eta \tag{13}$$

of vector bundles over $J^k_n(\xi)$ induced by φ_*. This is a k-th order linear differential operator from $V\xi$ to $V\eta$.

2 Differential Equations

2.1 Let M be a manifold, and fix $n \le \dim M$. Let $k > 0$. A *k-th order differential equation* on n-dimensional submanifolds of M is a subset $R \subseteq J^k_n(M)$. An n-dimensional submanifold $N \subseteq M$ is said to be a *solution* of R if $N^{(k)} \subseteq R$. We say that R is *smooth* if it is a smooth submanifold of $J^k_n(M)$.

A k-th order differential equation R is said to be *differentially closed* if for every smooth function $f : J^{k-1}_n(M) \to \mathbb{R}$ which (when pullbacked to $J^k_n(M)$) vanishes along R and every section X of $U^{(k-1)}$ over $J^k_n(M)$, we have that $X(f)$ also vanishes along R. In coordinates, this means that if R satisfies $f(x^i, u^a_I) = 0$ for some function f which only depends on u^a_I for $|I| \leq k-1$, then R also satisfies $D_j(x^i, u^a_I)$ for all $1 \leq j \leq n$, where D_j denotes the derivative with respect to x^j treating u^a_I as functions of x. A differential equation R is said to be *locally differentially closed* if $V \cap R$ is differentially closed for every open subspace $V \subseteq J^k_n(M)$. If an equation does not satisfy this, we may replace it by the largest locally differentially closed equation contained inside it, and its solutions would not change (although one may lose smoothness when doing this).

We say that a smooth differential equation R is *infinitesimally differentially closed* if for every $y \in R$ we have $U^{(k-1)}_y \subseteq \pi_{k,k-1*}T_yR$. The motivation for this last definition comes from the following

Proposition 2.1.1. *Any infinitesimally differentially closed equation $R \subseteq J^k_n(M)$ is also locally differentially closed. The converse is true provided that R is smooth and $\pi_{k,k-1}|_R$ has constant rank.*

Proof. Suppose that R is infinitesimally differentially closed. Let $V \subseteq J^k_n(M)$ be an open subspace, and $f : J^{k-1}_n(M) \to \mathbb{R}$ be a smooth function vanishing along $R \cap V$. Let $y \in R \cap V$. For each $X \in U^{(k-1)}_y$, we have that $X(f) = \mathrm{d}f(X)$. Now, $\mathrm{d}f$ vanishes on T_yR, and so it vanishes on $U^{(k-1)}_y$. This implies that $X(f) = 0$, so we conclude that R is locally differentially closed.

Now, assume that R is smooth and $U^{(k-1)}_y \nsubseteq \pi_{k,k-1*}T_yR$ for some $y \in R$. If $\pi_{k,k-1*}$ has constant rank near y, one may find a smooth function $f : J^{k-1}_n(M) \to \mathbb{R}$ which vanishes along R in a neighborhood of y, such that $\mathrm{d}f|_{U^{(k-1)}_y} \neq 0$. This implies that there exists $X \in U^{(k-1)}_y$ such that $X(f)(y) \neq 0$, and so R is not locally differentially closed. $\square$

Let $\varphi : J^k_n(M) \to M'$ be a differential operator, and $S \subseteq M'$ be a subset. Then $\varphi^{-1}(S)$ is a differential equation. We say that R is given (or presented) by $\varphi \in S$. The presentation is said to be *regular* if R is smooth, S is a submanifold of M', and the induced map φ_* from the normal bundle of R to the normal bundle of S is injective.

Observe that any equation may be presented in this way. Indeed, if $R \subseteq J^k_n(M)$, then R is given by $\mathrm{id}_k \in R$, for id_k the k-th order universal differential operator. Moreover, if R is smooth then this presentation is regular.

2.2 Let R be a smooth k-th order differential equation and let $l \geq 0$. The *l-th prolongation* of R is the $(k+l)$-th order equation $R^{(l)} = J^l_n(R) \cap J^{k+l}_n(M)$, where the intersection is taken inside $J^l_n(J^k_n(M))$. This equation is not necessarily smooth.

Observe that $R^{(l)}$ is given by $\mathrm{id}_k^{(l)} \in J_n^l(R)$. That is true more generally if the equation is given regularly by $\varphi \in S$, as we shall now prove. Here we assume as usual that our operators and their prolongations are globally defined. If not, one only recovers the intersection of $R^{(l)}$ with the domain of definition of $\varphi^{(l)}$. Before proving this, we need the following lemma

LEMMA 2.2.1. *Let $N \geq n \geq 0$ and $r \geq 0$ be natural numbers, and $g : \mathbb{R}^N \to \mathbb{R}^r$ be a smooth function such that $g^{-1}(\{0\})$ is a smooth submanifold of $\mathbb{R}^N$ and for every $y \in g^{-1}(\{0\})$ we have that $T_y g^{-1}(\{0\}) = g_*^{-1}(\{0\})$. Let $h : \mathbb{R}^n \to \mathbb{R}^N$ be a smooth function and $k \geq 0$. We have that the k-jet of gh at 0 vanishes if and only if the k-jet of h at 0 factors through $g^{-1}(\{0\})$.*

Proof. If the k-jet of h at 0 factors through $g^{-1}(\{0\})$ then the k-jet of gh at 0 vanishes by the chain rule.

Reciprocally, suppose that the k-jet of gh at 0 vanishes. Changing coordinates if necessary, we may assume that $g^{-1}(\{0\})$ is an hyperplane on $\mathbb{R}^N$. In this case, the k-jet of g at 0 factors through $g^{-1}(\{0\})$ if and only if $g_* D^j(g)(0) = 0$ for all $j \leq k$. By induction, we may assume that the $(k-1)$-jet of h at 0 factors through $g^{-1}(\{0\})$. If we apply the chain rule to $D^k(gh)$, the only surviving term is $g_* D^k(h)(0)$, which must therefore vanish. □

PROPOSITION 2.2.2. *Let $R \subseteq J_n^k(M)$ be a smooth equation given regularly by $\varphi \in S$. Then $R^{(l)}$ is given by $\varphi^{(l)} \in J_n^k(S)$.*

Proof. Let $y \in J_n^{k+l}(M)$ and choose functions $f^1, \ldots, f^r$ on M' defined around $\varphi(y_k)$ such that $f = (f^1, \ldots, f^r) : M' \to \mathbb{R}^r$ is a (locally defined) submersion and S coincides with the zero locus of f near $\varphi(y_k)$. Then R is defined near y_k by $f\varphi = 0$.

Let N be an n-dimensional submanifold of M passing through a point $q \in M$, and such that $i^{(k+l)}(q) = y$. The fact that $\varphi \in S$ is regular implies, by the lemma, that $i^{(k+l)}(q) \in R^{(l)}$ if and only if $f\varphi|_{N^{(k)}}$ vanishes at y_k up through order l. This is equivalent to $f|_{\varphi(N^{(k)})}$ vanishing at $\varphi(y_k)$ up through order l, and this happens if and only if $\varphi^{(l)}(y) \in J_n^l(S)$. □

We say that R is *integrable up through order l* if $R^{(j)}$ is smooth for $0 \leq j \leq l$, and the projections $R^{(j+1)} \to R^{(j)}$ are surjective submersions for $j < l$. In this case, proposition 2.2.2 may be strengthened to give

PROPOSITION 2.2.3. *Let $R \subseteq J_n^k(M)$ be a smooth equation given regularly by $\varphi \in S$, and suppose that R is integrable up through order l. Then $R^{(l)}$ is given regularly by $\varphi^{(l)} \in J_n^k(S)$.*

Proof. We have to check that the map induced by $\varphi_*^{(l)}$ between the normal bundles $\mathrm{Nor}R^{(l)}$ and $\mathrm{Nor}J_n^l(S)$ is injective. Assume that this is true for $l-1$. Let $\mathrm{Nor}_V R^{(l)} =$

$V\pi_{k+l,k+l-1}/(V\pi_{k+l,k+l-1}|_R)$ and $\mathrm{Nor}_V J^l_n(S) = V\pi_{l,l-1}/(V\pi_{l,l-1}|_{J^l_n(S)})$ be the vertical part of the normal bundles to $R^{(l)}$ and $J^l_n(S)$, respectively. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Nor}_V R^{(l)} & \longrightarrow & \mathrm{Nor} R^{(l)} & \longrightarrow & \mathrm{Nor} R^{(l-1)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Nor}_V J^l_n(S) & \longrightarrow & \mathrm{Nor} J^l_n(S) & \longrightarrow & \mathrm{Nor} J^{l-1}_n(S) & \longrightarrow & 0 \end{array} \qquad (14)$$

By the inductive hypothesis, the third vertical arrow is injective. Therefore, to prove that $\varphi^{(l)}_* : \mathrm{Nor}R^{(l)} \to \mathrm{Nor}J^l_n(S)$ is injective we only need to show that the corresponding map between the vertical normal bundles is injective.

We consider first the case $l = 1$. Let $y \in R^{(1)}$ and let $y' = \varphi^{(1)}y$. Let $v \in V_y\pi_{k+1,k} = S^{k+1}U^*_y \otimes Q_y$ be a vertical vector at y such that $\sigma^{(1)}_\varphi(v) \in V_{y'}\pi_{1,0}|_{J^1_n(S)}$. We want to show that $v \in V_y\pi_{k+1,k}|_{R^{(1)}}$. For each $X \in U_y$, we have $\sigma_\varphi(X \lrcorner v) = X \lrcorner \sigma^{(1)}_\varphi(v) \in T_{y'_0}S/U'_{y'}$. This implies that $\varphi_*(X \lrcorner v)$ is tangent to S, and therefore (since the presentation is regular) $X \lrcorner v \in V_{y_k}\pi_{k,k-1}|_R$. The fact that this is true for all X implies that $y + tv \in R^{(1)}$ for all $t \in \mathbb{R}$. Therefore $v \in V_y\pi_{k+1,k}|_{R^{(1)}}$, as we wanted.

We now let $l > 1$. As before, let $y \in R^{(l)}$ and $y' = \varphi^{(l)}(y)$. Let $v \in V_y\pi_{k+l,k+l-1}$ be a vertical vector at y such that $\sigma^{(l)}_\varphi(v) \in V_{y'}\pi_{l,l-1}|_{J^l_n(S)}$. Now, for any $t \in \mathbb{R}$ we have that $\varphi^{(l)}(y + tv) = y' + t\sigma^{(l)}(v)$, which belongs to $J^l_n(S)$ (using the fact that this is an affine subbundle of $J^l_n(M')$ over $J^{l-1}_n(S)$). Therefore $v \in V_y\pi_{k+l,k+l-1}|_{R^{(l)}}$, as we wanted. □

Corollary 2.2.4. *Let $R \subseteq J^k_n(M)$ be a differential equation which is integrable up through order l. Then $R^{(l)(m)} = R^{(l+m)}$ for all $m \geq 0$.*

Proof. We know that $R^{(l)}$ is given regularly by $\mathrm{id}^{(l)}_k \in J^l_n(R)$. Therefore, $R^{(l)(m)}$ is given by $\mathrm{id}^{(l)(m)}_k \in J^m_n(J^l_n(R)) \subseteq J^m_n(J^l_n(J^k_n(M)))$. Now, the image of $\mathrm{id}^{(l)(m)}_k$ is contained inside $J^{l+m}_n(J^k_n(M))$, and its co-restriction equals $\mathrm{id}^{(l+m)}_k$. Therefore $R^{(l)(m)}$ is given by $\mathrm{id}^{(l+m)}_k \in J^{l+m}_n(J^k_n(M)) \cap J^m_n(J^l_n(R))$. This space equals $J^{l+m}_n(R)$ by induction on l, and so $R^{(l)(m)}$ coincides with $R^{(l+m)}$. □

This corollary implies that one could also define $R^{(l)}$ inductively, taking first prolongation l times. To define $R^{(1)}$ one only needs knowledge of R, the restriction of the contact distribution $\mathcal{C}^k_n$ to R, and the vertical distribution $V\pi_{k,k-1}|_R$. Moreover, from this information one may also recover the solutions of R, as the integral submanifolds of $\mathcal{C}^k_n|_R$ transverse to the vertical. Therefore, one could develop the theory of differential equations inductively, and forget about the fact that R embeds inside a jet space. This is the point of view usually taken in the literature on exterior differential systems.

2.3 Let $R \subseteq J^k_n(M)$ be a smooth k-th order differential equation. The *principal symbol* of R is the (possibly singular) vector bundle

$$\mathcal{A}^k_R = V\pi_{k,k-1}|_R = \ker(\pi_{k,k-1*}|_{TR}) \subseteq S^kU^* \otimes Q \tag{15}$$

defined over R. The *sub-principal symbols* of R are the (possibly singular) vector bundles

$$\mathcal{A}^j_R = \ker(\pi_{j,j-1*}|_{\pi_{k,j*}TR}) \subseteq S^jU^* \otimes Q \tag{16}$$

defined for $1 \leq j \leq k$. We also set $\mathcal{A}^0_R = p_Q\pi_{k,0}TR$, where $p_Q : TM \to Q$ is the projection.

Let $l \geq 0$. The *l-th prolongation* of the principal symbol of R is the (possibly singular) vector bundle

$$\mathcal{A}^{k+l}_R = (S^lU^* \otimes A_R) \cap (S^{k+l}U^* \otimes Q) \tag{17}$$

where the intersection is taken inside $S^lU^* \otimes (S^kU^* \otimes Q)$. In other words, the fiber of $\mathcal{A}^{k+l}_R$ at a point $y \in R$ consists of those polynomials in $S^{k+l}U^*_y \otimes Q_y$ such that all their derivatives of order l belong to $\mathcal{A}^k_{R,y}$. If R is given regularly by $\varphi \in S$, then we have $\mathcal{A}^{k+l}_R = (\sigma^{(l)}_\varphi)^{-1}(S^lU^* \otimes (TS/U'))$ for $l \geq 0$.

Let $j \geq 0$. We say that R is *j-regular* if the dimension of the fibers of $\mathcal{A}^j_R$ is constant. In particular, observe that R is k-regular if and only if $\pi_{k,k-1}|_R$ has constant rank. Moreover, R is j-regular for all $0 \leq j \leq k$ if and only if $p_Q\pi_{k,0}|_R$ and $\pi_{k,j}|_R$ have constant rank, for all $0 \leq j < k$.

The *total prolongation* of the symbol of R is the (infinite dimensional) vector bundle

$$\mathcal{A}_R = \bigoplus_{j=0}^{\infty} \mathcal{A}^j_R \subseteq SU^* \otimes Q \tag{18}$$

Proposition 2.3.1. *Let $j_0 \geq 1$, and let $R \subseteq J^k_n(M)$ be an infinitesimally differentially closed equation which is j-regular for all $j_0 \leq j \leq k$. Then the image of the contraction map $U \otimes \mathcal{A}^j_R \to S^{j-1}U^* \otimes Q$ is contained in $\mathcal{A}^{j-1}_R$ for all $j \geq j_0$.*

Proof. This is true in degrees greater than k by definition. Let $j_0 \leq j \leq k$. Let $y \in R$, and $v \in \mathcal{A}^j_{R,y}$. We have to prove that $X \lrcorner v \in \mathcal{A}^{j-1}_R$ for each $X \in U_y$.

Let γ be a curve on R such that $\gamma(0) = y$ and $\partial_t(\pi_{k,j}\gamma)(0) = v$. By regularity, we may assume that $\pi_{k,j-1}\gamma$ is constant. Let $X^{(j-1)}_{\gamma(t)}$ be the lift of X to $U^{(j-1)}_{\gamma(t)} \subseteq T_{y_{j-1}}J^{j-1}_n(M)$. When $j \geq 2$, $X \lrcorner v \in V\pi_{j-1,j-2}$ may be computed, as $\partial_t(X^{(j-1)}_\gamma)(0)$ (in the case $j = 1$ one needs to project this vector to Q to obtain $X \lrcorner v$).

Now, consider the curve $\alpha = (\gamma, X^{(j-1)}_\gamma)$ on the pullback of $TJ^{j-1}_n(M)$ to $J^k_n(M)$. The fact that R is infinitesimally differentially closed implies that α is contained in $\pi_{k,j-1*}TR$ which, by regularity, is a smooth subbundle of $TJ^{j-1}_n(M)$ over $J^k_n(M)$. The derivative of α at 0 belongs to the fiber of $\pi_{k,j-1*}TR$ at y, and may be identified with the derivative of $X^{(j-1)}_\gamma$ at 0. This implies that $X \lrcorner v \in \mathcal{A}^{j-1}_R$, as we wanted. □

As a consequence of this proposition, if R is an infinitesimally differentially closed equation which is j-regular for $1 \le j \le k$ we have that $\mathcal{A}_R \subseteq SU^* \otimes Q$ is a bundle of SU^* subcomodules, and we may form the Spencer cohomology bundles $H^{q,j}(\mathcal{A}_R)$. Even if $\mathcal{A}_R$ is only k-regular, the dual of the Koszul complex still makes sense in some degrees, so the bundles $H^{q,j+q}(\mathcal{A}_R)$ may still be computed for $j \ge k$. Moreover, the bundle $H^{q,k+q-1}(\mathcal{A}_R)$ may be defined as the cohomology of the sequence

$$\mathcal{A}_R^k \otimes \Lambda^{q-1}U^* \xrightarrow{\delta^{q-1}} \mathcal{A}_R^{k-1} \otimes \Lambda^q U^* \xrightarrow{\delta^q} (S^{k-2}U^* \otimes Q) \otimes \Lambda^{q+1}U^* \tag{19}$$

The prolonged symbols govern the behavior of the prolongations of R, as follows

Proposition 2.3.2. *Let $R \subseteq J_n^k(M)$ be a smooth k-th order differential equation and let $l \ge 1$. The non-empty fibers of $\pi_{k+l,k+l-1}|_{R^{(l)}}$ are affine spaces modeled on $\mathcal{A}_R^{k+l}$.*

Proof. Consider first the case $l = 1$. Let $y \in R$, and let $z, \bar{z}$ be two points in the fiber of $\pi_{k+1,k}$ over y, with $z \in R^{(1)}$. Then $\bar{z} \in R^{(1)}$ if and only if $U_{\bar{z}}^{(k)}$ is tangent to R. As we already know that $U_z^{(k)}$ is tangent to R, we have that $\bar{z} \in R^{(1)}$ if and only if $X \lrcorner (\bar{z} - z) \in \mathcal{A}_R^k$ for all $X \in U_y$, which is the same as saying $\bar{z} - z \in \mathcal{A}_R^{k+1}$.

Now, let $l \ge 2$. The fiber of $R^{(l)}$ over a point $y \in R^{(l-1)}$ is the intersection of fibers of $J_n^{k+l}(M)$ and $J_n^l(R)$ over y. These are two affine subspaces of the fiber of $J_n^l(J_n^k(M))$ over y, modeled on $S^{k+l}U^* \otimes Q$ and $S^l U^* \otimes (TR/U^{(k)})$, so their intersection is an affine space modeled on $(S^{k+l}U^* \otimes Q) \cap (S^l U^* \otimes (TR/U^{(k)})) = \mathcal{A}_R^{k+l}$ (here we are implicitly using proposition 1.2.1 in the case of the universal differential operator, to conclude the compatibility of the affine structure on $J_n^{k+l}(M)$ and $J_n^l(J_n^k(M))$. ☐

Corollary 2.3.3. *If R is integrable up through order l, then it is j-regular for $k < j \le k + l$, and $R^{(m)} \to R^{(m-1)}$ is an affine bundle modeled on $\mathcal{A}_R^{k+m}$ for all $1 \le m \le l$. In particular, $\mathcal{A}_{R^{(l)}} = \mathcal{A}_R$.*

2.4 Let $\xi : E_\xi \to M$ be a fibered manifold. A *k-th order differential equation* on sections of ξ over n-dimensional submanifolds of M is a subset $R \subseteq J_n^k(\xi)$. We say that R is smooth if it is a smooth submanifold of $J_n^k(\xi)$ and $TR \cap T_n J_n^k(\xi)$ is a smooth vector bundle over $R \times_{J_n^k(M)} J_n^{k+1}(M)$. If R is smooth, then its l-th prolongation $R^{(l)}$ belongs to $J_n^{k+l}(\xi)$ so it is still a differential equation on sections of ξ.

Let $\varphi : J_n^k(\xi) \to E_\eta$ be a differential operator taking values in the fibered manifold $\eta : E_\eta \to J_n^k(M)$. Let $s : J_n^k(M) \to E_\eta$ be a section. We may define the k-th order differential equation $R \subseteq J_n^k(\xi)$ consisting of those $y \in J_n^k(\xi)$ such that $\varphi(y) = s\xi^{(k)}y$. Observe that R is given by $\varphi \in S$, where S is the image of s.

Now, assume that R is smooth. We say that R is infinitesimally differentially closed if $U_\xi^{(k-1)}|_R \subseteq \pi_{k,k-1*}TR$. One may define analogues to the definitions of local and global differential closedness, so that proposition 2.1.1 continues to hold. The (local) clausure of an equation makes sense only as a subset of $J_n^k(\xi)$ over $J_n^\infty(M) = \varprojlim J_n^k(M)$.

The principal and sub-principal symbols of R are now defined as

$$\mathcal{A}_R^j = \ker(\pi_{j,j-1*}|_{\pi_{k,j*}(V\xi^{(k)}|_R)}) \subseteq V\xi^{(j)} = S^jU^* \otimes V\xi \tag{20}$$

for $0 < j \leq k$. This coincides with what get from the previous definition if we interpret the maps $\pi_{k,j}$ and $\pi_{j,j-1}$ as being morphisms of fibered manifolds over $J_n^k(M)$. We also set $\mathcal{A}_R^0 = \pi_{k,0*}(V\xi^{(k)}|_R)$.

When $j \geq k$, we define $\mathcal{A}_R^j = (S^{j-k}U^* \otimes \mathcal{A}_R^k) \cap (S^jU^* \otimes V\xi)$. We say that R is j-regular if $\mathcal{A}_R^j$ is smooth. The total prolongation is again defined as $\mathcal{A}_R = \bigoplus_0^\infty \mathcal{A}_R^j \subseteq SU^* \otimes V\xi$. Propositions 2.3.1, 2.3.2 and corollary 2.3.3 continue to hold in this context when interpreted properly (in particular, the notion of integrability up through order l reads the same as before, only that $R^{(m)} \to R^{(m-1)}$ has to be considered over $J_n^{k+m}(M)$).

In the case when ξ is a vector bundle, a k-th order differential equation $R \subseteq J_n^k(\xi)$ is said to be *linear* if it is a (possibly singular) vector subbundle of $J_n^k(\xi)$ over $J_n^k(M)$. If R is smooth, then its l-th prolongation $R^{(l)}$ is again a linear differential equation . For $l \leq 0$, the sub-principal symbol $A_R^{(l)}$ may be identified with the vector bundle $\ker(\pi_{k+l,k+l-1}|_{\pi_{k,k+l}R}) \subseteq S^{k+l}U^* \otimes E_\xi$ over $J_n^k(M)$. Proposition 2.3.2 implies that for $l \geq 1$ we have $\ker(\pi_{k+l,k+l-1}|_{R^{(l)}}) = \mathcal{A}_R^{k+l}$ as vector bundles over $J_n^{k+l}(M)$.

2.5 Let $R \subseteq J_n^k(M)$ be a smooth k-th order differential equation, integrable up to first order. The *linearization* of R is the subbundle

$$\ell_R = TR/U^{(k)} \subseteq TJ_n^k(M)/U^{(k)} = J_n^k(Q \to J_n^1(M)) \tag{21}$$

where $J_n^k(Q \to J_n^1(M))$ is considered as a vector bundle over $R^{(1)}$. One should note that this is not a differential equation in the above sense, since it is only defined over $R^{(1)}$. However, if one assumes integrability of R, the linearization ℓ_R behaves as a linear differential equation, whose prolongations are only defined over $R^{(l)}$. If $i : N \to M$ is a solution of R, then the pullback $i^{(k+1)*}\ell_R$ is a k-th order differential equation on sections of the normal bundle of N, called the *linearization of R at N*.

3 Formal Integrability

3.1 Let M be a differentiable manifold and fix $n \leq \dim M$. Let $k > 0$, and let $R \subseteq J_n^k(M)$ be a k-th order differential equation. We say that R is *formally integrable* if it is integrable up through order l for all $l > 0$. In this case, any element of $R^{(l)}$ lying over $q \in M$ may be extended to an element of $R^{(\infty)} = \varprojlim R^{(m)}$ over q, which may be thought of as a formal solution to the equation at q. This is still not an actual solution of the equation, however in the analytic category there is the following

THEOREM 3.1.1. *Let M be an analytic manifold, and $R \subseteq J_n^k(M)$ be a formally integrable analytic differential equation. Then for every $l > 0$ and $y \in R^{(l)}$, there is an analytic solution N of R such that $y \in N^{(k+l)}$.*

We refer the reader to Goldschmidt[5] for a proof, which depends on the "δ-Poincaré estimate" of Spencer. See Ehrenpreis, Guillemin, Sternberg[2] or Sweeney[13] for a discussion and proof of the estimate. One may also prove this theorem using the Cartan-Kahler theorem in the theory of exterior differential systems, which ultimately depends on the Cauchy-Kowalevski existence theorem for analytic partial differential equations, see [1].

In the fibered case, we have the following

THEOREM 3.1.2. *Let $\xi : E_\xi \to M$ be an analytic fibered manifold, and $R \subseteq J^k_n(\xi)$ be an analytic formally integrable differential equation on sections. Let $i : N \to M$ be an n-dimensional submanifold of M passing through a point $q \in M$. Then, for every $l > 0$ and $y \in R^{(l)}$ such that $\xi^{(k+l)}(y) = i^{(k+l)}(q)$, there exists a section s of $\xi|_N$ defined near q, such that $s^{(k+l)}(q) = y$.*

This follows from the above result, applied to the pullbacked equation $i^{(k)*}R \subseteq J^k(\xi|_N)$.

3.2 Let M be a manifold and fix $n \leq \dim M$. Let $\check{J}^{k+1}_n(M)$ be the bundle of sesqui-holonomic jets of order $k+1$. Recall that this is the bundle over $J^k_n(M)$ whose fiber over a point y consists of the planes $\Pi \subseteq \mathcal{C}^k_{n,y}$ giving a splitting of

$$0 \to V_y\pi_{k,k-1} \to \mathcal{C}^k_{n,y} \to U^{(k-1)}_y \to 0 \tag{22}$$

This is an affine bundle modeled on $U^* \otimes (S^kU^* \otimes Q)$. The form $\overline{[\cdot,\cdot]}$ defined in I.3.2 may be restricted to each sesqui-holonomic jet, so we have a map

$$C : \check{J}^{k+1}_n(M) \to (S^{k-1}U^* \otimes Q) \otimes \Lambda^2 U^* \tag{23}$$

Of course, this map also has a dual description in terms of exterior differentiation of contact forms.

PROPOSITION 3.2.1. *The map C is an affine bundle map modeled on minus the first Spencer coboundary map*

$$-\delta^1 : (S^kU^* \otimes Q) \otimes U^* \to (S^{k-1}U^* \otimes Q) \otimes \Lambda^2 U^* \tag{24}$$

Moreover, we have that $\delta^2 C = 0$, where

$$\delta^2 : (S^{k-1}U^* \otimes Q) \otimes \Lambda^2 U^* \to (S^{k-2}U^* \otimes Q) \otimes \Lambda^3 U^* \tag{25}$$

is the second Spencer coboundary map.

Proof. Let $y \in J^k_n(M)$, and $\Pi \in \check{J}^{k+1}_n(M)_y$. We want to compute $C(\Pi+\Delta) - C(\Pi)$ for each $\Delta \in (S^k U^* \otimes Q)_y \otimes U^*_y$.

Let $X, Y \in U_y$, and denote by X_Π, Y_Π their lifts to Π. By definition, we have $C(\Pi)(X,Y) = \overline{[X_\Pi, Y_\Pi]}$. Now,

$$C(\Pi+\Delta)(X,Y) = \overline{[X_\Pi + \Delta(X), Y_\Pi + \Delta(Y)]} \tag{26}$$
$$= C(\Pi)(X,Y) + \overline{[\Delta(X), Y_\Pi]} - \overline{[\Delta(Y), X_\Pi]} \tag{27}$$

Now, by proposition I.3.2.1, $\overline{[\Delta(X), Y_\Pi]} - \overline{[\Delta(Y), X_\Pi]} = -\delta^1(\Delta)(X,Y)$, which proves the first part of the proposition.

For the second part, let Π' be an integral element of the contact system at y. We know that $C(\Pi') = 0$, and therefore $C(\Pi) = \delta^1(\Pi - \Pi')$, which implies that $C(\Pi)$ is closed. □

3.3 We are now ready to describe the obstruction for an equation to be integrable to first order. Proceeding inductively, we will have a series of obstructions for an equation to be formally integrable.

Let $R \subseteq J^k_n(M)$ be a k-regular infinitesimally differentially closed equation. Let $R^{\langle 1\rangle}$ be the subspace of $\check{J}^{k+1}_n(M)$ consisting of those planes tangent to R. The projection $R^{\langle 1\rangle} \to R$ is an affine subbundle of $\check{J}^{k+1}_n(M)$ modeled on $\mathcal{A}^k_R \otimes U^*$. Moreover, the map C restricts to give a map

$$C : R^{\langle 1\rangle} \to \mathcal{A}^{k-1}_R \otimes \Lambda^2 U^* \tag{28}$$

By proposition 3.2.1, this descends to a well defined section

$$\kappa_R : R \to H^{2,k+1}(\mathcal{A}_R) \tag{29}$$

called the *curvature* of R. This is the obstruction to integrability that we needed:

PROPOSITION 3.3.1. *Let $R \subseteq J^k_n(M)$ be a k-regular infinitesimally differentially closed equation. Then R is integrable to first order if and only if it is $(k+1)$-regular and the curvature κ_R vanishes.*

Proof. If R is integrable to first order, then $\mathcal{A}^{k+1}_R = V\pi_{k+1,k}|_{R^{(1)}}$ is smooth. Moreover, for every $y \in R$ there exists a plane $\Pi \in R^{(1)}$ which is integral for the contact system on $J^k_n(M)$ (namely, $\Pi = U^{(k)}_z$ for any $z \in R^{(1)}$ over y). This plane satisfies $C(\Pi) = 0$, and so the curvature vanishes.

Conversely, suppose that κ_R vanishes and that R is $(k+1)$-regular. Let s be a smooth section of $R^{\langle 1\rangle}$. Then Cs is a smooth section of $(S^{k-1}U^* \otimes Q) \otimes \Lambda^2 U^*$, which belongs to $\delta^1(\mathcal{A}^k_R \otimes U^*)$ since the curvature vanishes. One may therefore replace s by a section s' such that $C(s') = 0$. We claim that we may do this smoothly.

By the polynomial Poincaré lemma, the kernel of $\delta^1|_{\mathcal{A}_R^k\otimes U^*}$ is contained in $\delta^0(S^kU^*\otimes Q)$. However, by definition of $\mathcal{A}_R^{k+1}$ one has that $(\delta^0)^{-1}(\mathcal{A}_R^k\otimes U^*)=\mathcal{A}_R^{k+1}$, and so we have an exact sequence

$$0\to\mathcal{A}_R^{k+1}\xrightarrow{\delta^0}\mathcal{A}_R^k\otimes U^*\xrightarrow{\delta^1}\mathcal{A}_R^{k-1}\otimes\Lambda^2U^* \tag{30}$$

Since $\mathcal{A}_R^{k+1}$ and $\mathcal{A}_R^k$ are smooth vector bundles, we see that $\delta^1|_{\mathcal{A}_R^k\otimes U^*}$ has constant rank. Therefore, there is a smooth section Δ of $\mathcal{A}_R^k\otimes U^*$ such that $\delta^1(\Delta)=C(s)$. Now, $s'=s+\Delta$ is a smooth section of $R^{(1)}$ over R. The existence of such a section, together with the fact that the fibers of $R^{(1)}$ are affine spaces modeled on $\mathcal{A}_R^{k+1}$ (which is smooth), implies that $R^{(1)}\to R$ is a (surjective) smooth submersion. □

Corollary 3.3.2. *Let $R\subseteq J_n^k(M)$ be a k-regular infinitesimally differentially closed equation. If $H^{2,j}(\mathcal{A}_R)=0$ for $j\geq k+1$, we have that R is formally integrable.*

Proof. For each $j\geq k+1$, consider the following sequence of bundles over R

$$0\to\mathcal{A}_R^{j}\xrightarrow{\delta^0}\mathcal{A}_R^{j-1}\otimes U^*\xrightarrow{\delta^1}\mathcal{A}_R^{j-2}\otimes\Lambda^2U^*\xrightarrow{\delta^2}(S^{j-3}U^*\otimes Q)\otimes\Lambda^3U^* \tag{31}$$

This is exact at $\mathcal{A}_R^{j-2}\otimes\Lambda^2U^*$ by hypothesis, and at $\mathcal{A}^{j-1}\otimes U^*$ as a consequence of the polynomial Poincaré lemma together with the definition of $\mathcal{A}_R^j$.

We claim that R is j-regular for all $j\geq k$. Assume that we know this for $i<j$. Observe that the rank of $\delta^1|_{\mathcal{A}_R^{j-1}\otimes U^*}$ is lower semi-continuous. Since the image of δ^1 coincides with the kernel of δ^2, it must also be upper semi-continuous, and so $\delta^1|_{\mathcal{A}_R^{j-1}\otimes U^*}$ must have constant rank. Since its kernel equals $\mathcal{A}_R^j$, it follows that R is also j-regular, completing the inductive step.

The corollary now follows from proposition 3.3.1 together with the fact that, if R is integrable up through some order $l\geq 0$, the curvature $\kappa_{R^{(l)}}$ belongs to $H^{2,k+l+1}(\mathcal{A}_R)$ □

3.4 As a first application, we shall see how the Frobenius integrability condition arises from this point of view. Let M be a manifold, and $\mathcal{C}$ be a distribution on M. Let $n\leq\operatorname{rank}\mathcal{C}$, and consider the first order differential equation $R\subseteq J_n^1(M)$ consisting of those n-dimensional planes tangent to $\mathcal{C}$. We have $\mathcal{A}_R^1=U^*\otimes\mathcal{C}/U$, and therefore $\mathcal{A}_R^j=S^jU^*\otimes\mathcal{C}/U$ for $j\geq 1$. From the polynomial Poincaré lemma we get $H^{2,2}(\mathcal{A}_R)=\Lambda^2U^*\otimes TM/\mathcal{C}$ and $H^{2,j}(\mathcal{A}_R)=0$ for $j\geq 3$.

We want to compute $\kappa_R(y)\in(TM/\mathcal{C})_{y_0}\otimes\Lambda^2U_y^*$ for $y\in R$. Let $X,Y\in U_y$. Let s be a smooth section of R such that $s(y_0)=y$. Extend X,Y to vector fields $\bar X,\bar Y$ on M, tangent to the distribution $q\mapsto U_{sq}$. Let $\Pi=s_*U_y$. Observe that $s_*\bar X,s_*\bar Y$ may be extended to contact vector fields on R, so we have

$$C(\Pi)(X,Y)=\overline{[s_*\bar X,s_*\bar Y]}_y=\overline{[\bar X,\bar Y]}_{y_0} \tag{32}$$

where $\overline{[\bar{X},\bar{Y}]_{y_0}}$ denotes the projection of $[\bar{X},\bar{Y}]_{y_0}$ to Q_y. Therefore, $\kappa_R(y)(X,Y)$ equals the class of $[\bar{X},\bar{Y}]_{y_0}$ in $(TM/\mathcal{C})_{y_0}$, for any choice of section s and extensions $\bar{X},\bar{Y}$. In particular, when $n = \operatorname{rank}\mathcal{C}$ the vanishing of the curvature is equivalent to the Frobenius integrability condition on $\mathcal{C}$.

In the case where we take $\mathcal{C}$ to be the contact distribution $\mathcal{C}^k_n$ on the jet space $J^k_n(M)$, the elements of R transverse to the vertical such that the curvature vanishes are exactly the integral elements of the contact system. Hence, when we restrict this equation to (an open subset of) the vanishing locus of the curvature, we get a formally integrable equation. This does not hold for a general distribution $\mathcal{C}$. Even though $H^{2,j}(\mathcal{A}_R) = 0$ for $j \geq 3$, when we restrict R the symbol changes and new integrability conditions may arise. For instance, this is the case when we take our distribution to be the contact distribution on a k-th order equation which is not formally integrable.

3.5 Let $\xi : E_\xi \to M$ and $\eta : E_\eta \to M$ be fibered manifolds, and $\varphi : J^k(\xi) \to E_\eta$ be a k-th order differential operator. Let $s : M \to E_\eta$ be a section of η. Let $R \subseteq J^k(\xi)$ be the equation consisting of those k-jets y such that $\varphi(y) = s(\xi(y))$. Assume that

$$\varphi \times \pi_{k,k-1} : J^k(\xi) \to E_\eta \times_M J^{k-1}(\xi) \tag{33}$$

is a surjective submersion, so that R is smooth and $R \to J^{k-1}(\xi)$ is a surjective submersion. We are going to construct the curvature κ_R in an alternative way, making use of the fact that R is given by a differential operator.

Let $y \in R$ and $q = \xi\pi_{k,0}y$. For each z in the fiber of $J^{k+1}(\xi)$ over y, consider the element $\bar{C}(z) = \varphi^{(1)}(z) - s^{(1)}(q) \in T^*M \otimes V\eta$. The class of $\bar{C}(z)$ in $T^*M \otimes V\eta/\mathrm{im}\sigma^{(1)}_\varphi$ does not depend on z, and is denoted $\bar{\kappa}_R(y)$. Observe that

$$0 \to \mathcal{A}_R \to ST^*M \otimes V\xi \xrightarrow{\sigma^\bullet_\varphi} S^{[-k]}T^*M \otimes V\eta \tag{34}$$

is the beginning of a minimal resolution for $\mathcal{A}_R$, where $\sigma^\bullet_\varphi$ denotes the sum of the prolonged symbols of φ. Therefore $\bar{\kappa}_R$ is a section of $T^*M \otimes V\eta/\mathrm{im}\sigma^{(1)}_\varphi = H^{2,k+1}(\mathcal{A}_R)$. Since $R^{(1)}$ is given by those z such that $\varphi^{(1)}(z) = s^{(1)}(q)$, we see that R is integrable to the first order if and only if $\bar{\kappa}_R$ vanishes. In fact, we have the following

Proposition 3.5.1. *The curvature κ_R of R coincides with $\bar{\kappa}_R$.*

Proof. Observe that to construct κ_R we used the description of the Spencer cohomology via the dual to the Koszul complex, while for $\bar{\kappa}_R$ we used a minimal resolution of $\mathcal{A}_R$. Therefore, to show that both curvatures coincide, we will have to pass from one description of the cohomology to the other, which ultimately depends on the commutativity of the Cotor. The relevant diagram is the following

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{A}_R^{k-1} \otimes \Lambda^2 T^*M & \longrightarrow & (S^{k-1}T^*M \otimes V\xi) \otimes \Lambda^2 T^*M & & \\
 & & \big\uparrow & & \big\uparrow{\scriptstyle \delta^1} & & \\
0 & \longrightarrow & \mathcal{A}_R^{k} \otimes T^*M & \longrightarrow & (S^{k}T^*M \otimes V\xi) \otimes T^*M & \xrightarrow{\sigma_\varphi \otimes 1_{T^*M}} & V\eta \otimes T^*M \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle \delta^0} \\
0 & \longrightarrow & \mathcal{A}_R^{k+1} & \longrightarrow & S^{k+1}T^*M \otimes V\xi & \xrightarrow{\sigma_\varphi^{(1)}} & T^*M \otimes V\eta
\end{array} \tag{35}$$

Let $y \in R$ and z be a point in the fiber of $J^{k+1}(\xi)$ over y. Let $\Pi \subseteq \mathcal{C}_n^k|_{R,y}$ be an n-dimensional plane giving a splitting of

$$0 \to \mathcal{A}_R^k \to \mathcal{C}_n^k|_R \to U^{(k-1)} \to 0 \tag{36}$$

at y. Let $\Pi' = U_z^{(k)}$. This gives a splitting of

$$0 \to S^kT^*M \otimes V\xi \to \mathcal{C}_n^k \to U^{(k-1)} \to 0 \tag{37}$$

at y. Therefore, $\Pi' - \Pi$ defines an element of $(S^kT^*M \otimes V\xi) \otimes T^*M$, and we have that $(\sigma_\varphi \otimes 1_{T^*M})(\Pi' - \Pi) = \delta^0(\bar{C}(z))$.

We now have that $\delta^1(\Pi' - \Pi) \in \mathcal{A}_R^{k-1} \otimes \Lambda^2 T^*M$ is closed, and its class in the cohomology equals $\bar{\kappa}_R(y)$. By proposition 3.2.1, $\delta^1(\Pi' - \Pi) = C(\Pi) - C(\Pi')$. By construction, Π' is an integral element of the contact system on $J_n^k(\xi)$, so $C(\Pi')$ vanishes. Therefore $\delta^1(\Pi' - \Pi) = C(\Pi)$, whose class is, by definition, $\kappa_R(y)$. □

The curvatures of the prolongations of R may be constructed in a similar way. Assume that R is integrable up through order l for some $l \geq 0$. Let $y \in R^{(l)}$. For each z in the fiber of $J^{k+l+1}(\xi)$ over y, consider the element $\bar{C}(z) = \varphi^{(k+l+1)}(z) - s^{(k+l+1)}(q)$ in $S^{l+1}T^*M \otimes V\eta$. Denote by $\bar{\kappa}_{R^{(l)}}(y)$ the class of $\bar{C}(z)$ modulo $\operatorname{im} \sigma_\varphi^{(k+l+1)}$.

Proposition 3.5.2. *We have that $\bar{\kappa}_{R^{(l)}}$ vanishes under contraction by sections of TM, so $\bar{\kappa}_{R^{(l)}}$ is a section of $H^{2,k+l+1}(\mathcal{A}_R)$. Moreover, $\bar{\kappa}_{R^{(l)}}$ coincides with the curvature $\kappa_{R^{(l)}}$.*

The proof of this goes along the same lines as the one given for proposition 3.5.1. The fact that $\bar{\kappa}_{R^{(l)}}$ is closed under contraction (which in the case $l = 0$ was obvious), now follows from the fact that $\delta^0(\bar{C}(z))$ belongs to the image of $\sigma_\varphi^{(l)} \otimes 1_{T^*M}$.

4 Initial Value Problems

4.1 Let M be a manifold and fix $n \leq \dim M$. Let $k > 0$. Denote by $J_n^k(M)^{(1,0)}$ the bundle over $J_n^k(M)$ whose fiber over a point y consists of the $(n-1)$-dimensional subspaces $\Pi_{n-1} \subseteq \mathcal{C}^k_{n,y}$ transverse to $V\pi_{k,k-1}$, and such that $\overline{[\cdot,\cdot]}|_{\Pi_{n-1}}$ vanishes. Let $J^k_n(M)^{(1,1)} \subseteq J^1_{n-1}(J^k_n(M)) \times_{J^k_n(M)} J^{k+1}_n(M)$ be the collection of pairs (Π_{n-1}, w) such that $\Pi_{n-1} \subseteq U^{(k)}_w$. Notice that the image of the projection $J^k_n(M)^{(1,1)} \to J^1_{n-1}(J^k_n(M))$ is contained in $J^k_n(M)^{(1,0)}$. Denote by $\alpha_1 : J^k_n(M)^{(1,1)} \to J^k_n(M)^{(1,0)}$ and $\beta_1 : J^k_n(M)^{(1,1)} \to J^{k+1}_n(M)$ the projections. Observe that $J^k_n(M)^{(1,1)}$ is smooth and β_1 is a surjective submersion.

Let $G_{n-1}(U)$ be the Grassmannian of hyperplanes of the universal bundle on $J^k_n(M)$. Let U_{n-1} be the universal bundle on $G_{n-1}(U)$. Denote by $(SU^* \otimes Q)|_{U_{n-1}}$ the bundle $SU^* \otimes Q$ over $G_{n-1}(U)$, considered as a bundle of SU^*_{n-1} comodules. The dual bundle is a bundle of free SU_{n-1} modules, and in particular $(SU^* \otimes Q)|_{U_{n-1}}$ is acyclic. Observe that we have a map $J^k_n(M)^{(1,0)} \to G_{n-1}(U)$ over $J^k_n(M)$, sending each plane Π_{n-1} in the fiber of $J^k_n(M)^{(1,0)}$ over y to its projection to U_y.

Proposition 4.1.1. *The projection $J^k_n(M)^{(1,0)} \to G_{n-1}(U)$ is a smooth affine bundle modeled on the kernel of the first Spencer coboundary map*

$$(S^kU^* \otimes Q)|_{U_{n-1}} \otimes U^*_{n-1} \xrightarrow{\delta^1} (S^{k-1}U^* \otimes Q)|_{U_{n-1}} \otimes \Lambda^2 U^*_{n-1} \tag{38}$$

Proof. Let $Y \to G_{n-1}(U)$ be the bundle whose fiber over $E \in G_{n-1}(U_y)$ consists of those $(n-1)$-dimensional subspaces of $\mathcal{C}^k_{n,y}$ transverse to $V_y\pi_{k,k-1}$, and whose projection to U_y equals E. This is a smooth affine bundle modeled on $V\pi_{k,k-1} \otimes U^*_{n-1}$. The form $\overline{[\cdot,\cdot]}$ induces, by restriction, a map

$$C_{n-1} : Y \to (S^{k-1}U^* \otimes Q) \otimes \Lambda^2 U^*_{n-1} \tag{39}$$

over Y, whose vanishing locus coincides with $J^k_n(M)^{(1,0)}$. As in propostion 3.2.1, this is seen to be an affine bundle map modeled on minus the map (38). This implies that $J^k_n(M)^{(1,0)} \to G_{n-1}(U)$ is an affine bundle modeled on the kernel of δ^1. The fact that (38) has constant rank implies that any smooth section of Y may be transformed into a smooth section with image inside $J^k_n(M)^{(1,0)}$. This, together with the smoothness of the kernel of (38), implies that $J^k_n(M)^{(1,0)} \to G_{n-1}(U)$ is a smooth affine bundle, as we wanted. □

The following proposition gives, in particular, an alternative characterization of the elements of $J^k_n(M)^{(1,0)}$, as those planes which may be extended to an integral element of the contact system.

Proposition 4.1.2. *The projection $\alpha_1 : J^k_n(M)^{(1,1)} \to J^k_n(M)^{(1,0)}$ is a (smooth) affine bundle modeled on $H^{0,k+1}((SU^* \otimes Q)|_{U_{n-1}}) \otimes (U/U_{n-1})^*$.*

Proof. Let Π_{n-1} be an element in the fiber of $J^k_n(M)^{(1,0)}$ over $y \in J^k_n(M)$, and extend it to an n-dimensional subspace $\Pi \subseteq \mathcal{C}^k_{n,y}$ complementing $V\pi_{k,k-1}$. Let X be the projection of Π_{n-1} to U_y, and let $Q_X = U_y/X$. Observe that the form $C(\Pi)$ vanishes when restricted to X, and so it may be considered as an element of $(S^{k-1}U^* \otimes Q)_y \otimes X^* \otimes Q_X^*$.

Consider the following exact sequence

$$(S^k U^* \otimes Q)_y \otimes U_y^* \xrightarrow{\delta^1} (S^{k-1}U^* \otimes Q)_y \otimes \Lambda^2 U_y^* \xrightarrow{\delta^2} (S^{k-2}U^* \otimes Q)_y \otimes \Lambda^3 U_y^* \tag{40}$$

If we consider only forms which vanish when restricted to X, we have

$$(S^k U^* \otimes Q)_y \otimes Q_X^* \xrightarrow{\delta^1} (S^{k-1}U^* \otimes Q)_y \otimes X^* \otimes Q_X^* \xrightarrow{\delta^2} (S^{k-2}U^* \otimes Q)_y \otimes \Lambda^2 X^* \otimes Q_X^* \tag{41}$$

This computes the space $H^{1,k}((SU^* \otimes Q)|_X) \otimes Q_X^*$ which vanishes, and so the sequence (41) is exact. Therefore, there exists $\Delta \in (S^k U^* \otimes Q)_y \otimes Q_X^*$ such that $\delta^1(\Delta) = C(\Pi)$. The plane $\Pi + \Delta$ is then an integral element of the contact system contained Π_{n-1}. Since Π_{n-1} was arbitrary, the surjectivity of α_1 follows.

If Π, Π' are two planes containing Π_{n-1}, we have that $\Pi - \Pi'$ belongs to $(S^k U^* \otimes Q)_y \otimes Q_X$. If Π' is an integral element of the contact system, the same is true for Π if and only if $\Pi - \Pi'$ is δ^1 closed. The kernel of δ^1 is $H^{0,k+1}((SU^* \otimes Q)|_{U_{n-1}}) \otimes (U/U_{n-1})^*$, and so we have the desired affine structure on the fibers of α_1. The existence of smooth sections for α_1 follows from the fact that δ^1 has constant rank. □

Proposition 4.1.3. *The first order differential equation $J_n^k(M)^{(1,0)} \subseteq J_{n-1}^1(J_n^k(M))$ is formally integrable.*

Proof. By homogeneity, $J_n^k(M)^{(1,0)}$ is j-regular for all j. Let Π_{n-1} be a point in the fiber of $J_n^k(M)^{(1,0)}$ over $y \in J_n^k(M)$. By the previous proposition, we may extend Π_{n-1} to an integral element of the contact system at y, which in turn may be extended to an integral submanifold $N^{(k)}$ of $\mathcal{C}_n^k$. The plane Π_{n-1} may be extended to an $(n-1)$-dimensional submanifold of N. The 2-jet of this submanifold is an element in the fiber of the first prolongation of $J_n^k(M)^{(1,0)}$ over Π_{n-1}. Since Π_{n-1} was arbitrary, we see that $J_n^k(M)^{(1,0)}$ is integrable to first order.

Let $L = \Pi/\Pi_{n-1}$. Let A be the principal symbol of $J_n^k(M)^{(1,0)}$ at Π_{n-1}. This is a first order tableau contained in

$$\Pi_{n-1}^* \otimes \mathcal{C}_{n,y}^k/\Pi_{n-1} = (\Pi_{n-1}^* \otimes (S^k U^* \otimes Q)_y) \oplus (\Pi_{n-1}^* \otimes L) \tag{42}$$

By proposition 4.1.1, we have

$$A = \ker \delta^1 \oplus (\Pi_{n-1}^* \otimes L) \tag{43}$$

where $\delta^1 : \Pi_{n-1}^* \otimes (S^k U^* \otimes Q)_y \to (S^{k-1}U^* \otimes Q)_y \otimes \Lambda^2 \Pi_{n-1}^*$ is the first Spencer coboundary. Therefore, the cohomology $H^2(A)$ of the first order tableau A equals $H^2(\ker \delta^1)$.

Observe that $(S^k U^* \otimes Q)_y$ may be decomposed as

$$\bigoplus_{j=0}^{k} (S^{k-j} L^* \otimes Q_y) \otimes S^j \Pi_{n-1}^* \tag{44}$$

By the polynomial Poincaré lemma, we have

$$\ker \delta^1 = \bigoplus_{j=0}^{k} (S^{k-j}L^* \otimes Q_y) \otimes S^{j+1}\Pi^*_{n-1} \tag{45}$$

and so the $S\Pi^*_{n-1}$-comodule associated to $\ker \delta^1$ is

$$\bigoplus_{j=0}^{k} (S^{k-j}L^* \otimes Q_y) \otimes S^{\geq j}\Pi^*_{n-1} \tag{46}$$

where $(S^{k-j}L^* \otimes Q_y) \otimes S^j\Pi^*_{n-1}$ is taken to have degree 0. From the following exact sequence

$$0 \to S^{j-1}\Pi^*_{n-1} \to S^{\geq j-1}\Pi^*_{n-1} \to S^{\geq j}\Pi^*_{n-1} \to 0 \tag{47}$$

one may see, by induction, that $S^{\geq j}\Pi^*_{n-1}$ is j-involutive for all $j \geq 0$ (where $S^j\Pi^*_{n-1}$ is taken to have degree j). From this, we have that (46) is a 1-involutive comodule. The proposition now follows, since one has $H^{2,j}(A) = 0$ for $j > 2$, and so the higher order obstructions to the integrability vanish. □

4.2 For each $l \geq 1$, let $J^k_n(M)^{(l,0)}$ be the $(l-1)$-th prolongation of $J^k_n(M)^{(1,0)}$. We also set $J^k_n(M)^{(0,0)} = J^k_n(M)$. We denote by $\pi_{l,m}$ both the projection $J^k_n(M)^{(l,0)} \to J^k_n(M)^{(m,0)}$ and $J^l_n(M) \to J^m_n(M)$, since it will be clear by context which one we are considering.

Let $J^k_n(M)^{(l,l)}$ be the subset of $J^l_{n-1}(J^k_n(M)) \times_{J^k_n(M)} J^{k+l}_n(M)$ consisting of those pairs (z, w) with z contained in w (where we interpret w as a l-jet of a submanifold on $J^k_n(M)$). Observe that we have $J^k_n(M)^{(l,l)} \subseteq J^k_n(M)^{(l,0)} \times_{J^k_n(M)} J^{k+l}_n(M)$. Let $\alpha_l : J^k_n(M)^{(l,l)} \to J^k_n(M)^{(l,0)}$ and $\beta_l : J^k_n(M)^{(l,l)} \to J^{k+l}_n(M)$ be the projections.

PROPOSITION 4.2.1. *The space $J^k_n(M)^{(l,l)}$ is a smooth manifold, and α_l, β_l are surjective submersions.*

Proof. The fact that $J^k_n(M)^{(l,l)}$ is smooth and β_l is a surjective submersion is evident in coordinates. One may also use coordinates to show that α_l is a surjective submersion, however we shall prove this using an intrinsic argument which contains the basic idea of our approach to solving the initial value problem.

Assume that α_j is a surjective submersion for $j < l$. Let $z \in J^k_n(M)^{(l,0)}$ and set $y = \pi_{l,0}z$. Let N_{n-1} be an $(n-1)$-dimensional submanifold of $J^k_n(M)$, passing through y, and whose l-jet at y equals z. Denote by $i_{n-1} : N_{n-1} \to J^k_n(M)$ the inclusion. By proposition 4.1.2, there exists a section $s : N_{n-1} \to J^{k+1}_n(M)$ such that the induced section

$$N^{(1)}_{n-1} \to J^1_{n-1}(J^k_n(M)) \times_{J^k_n(M)} J^{k+1}_n(M) \tag{48}$$

is tangent to $J^k_n(M)^{(1,1)}$ with order $(l-1)$ at $i^{(1)}_{n-1}(y)$. Equivalently, the l-jet of $s(N_{n-1})$ at $s(y)$ is integral to the contact distribution $\mathcal{C}^{k+1}_n$. This implies that the $(l-1)$-jet of $s(N_{n-1})$ at $s(y)$ belongs to $J^{k+1}_n(M)^{(l-1,0)}$. By induction, there exists a $(k+l)$-jet $w \in J^{k+l}_n(M)$ such that the $(l-1)$-jet of $s(N_{n-1})$ at $s(y)$ is contained in $w \in J^{k+l}_n(M) \subseteq J^{l-1}_n(J^{k+1}_n(M))$. This implies that $z = i^{(l)}_{n-1}(y)$ is contained in w. Since z was arbitrary, we have that α_l is surjective. Moreover, by induction one may take w to depend smoothly on z, and so one has that α_l is a submersion. □

4.3 Let M be a manifold and $n \leq \dim M$. Let $R \subseteq J^k_n(M)$ be a smooth k-th order equation. A subspace $\Pi \subseteq \mathcal{C}^k_n|_{R,y}$ of the contact distribution restricted to R at a point y is said to be *generic* if it is trasverse to $V\pi_{k,k-1}$ and its projection to U_y is a generic subspace for the k-th order tableau $A^k_{R,y}$. The *Cartan characters* $s_1, \ldots, s_n$ of R are the functions on R such that $s_j(y)$ is the j-th character of $\mathcal{A}^k_{R,y}$. We say that R is *involutive* if it is integrable to first order and $\mathcal{A}^k_R$ is a bundle of involutive tableaux. The involutivity of $\mathcal{A}^k_R$ is equivalent to the k-involutivity of $\mathcal{A}_R$, and therefore, by corollary 3.3.2, involutive equations are formally integrable (provided that the Cartan characters are constant). The following proposition is a weak version of the Cartan-Kuranishi prolongation theorem.

Proposition 4.3.1. *Let R be a k-th order formally integrable differential equation. We have that $R^{(l)}$ is involutive for l large enough.*

Proof. Consider the smooth bundle of $(k+1)$-th order tableaux $\mathcal{A}^{k+1}_R$. From proposition II.1.4.1 (specifically, the observation that the bound may be taken to only depend on the Hilbert function), we have that $\mathcal{A}^{k+l}_R$ is involutive for l large enough, as we needed. □

4.4 Let R be a smooth k-th order differential equation. Let $R^{(1,0)}$ be the subset of $J^k_n(M)^{(1,0)}$ consisting of panes Π_{n-1} tangent to R and generic. This is a first order equation on $(n-1)$-dimensional submanifolds of R.

Let $R^{(1,1)} \subseteq J^k_n(M)^{(1,1)}$ be the collection of pairs $(\Pi_{n-1}, \Pi) \in R^{(1,0)} \times_R R^{(1)}$ such that $\Pi_{n-1} \subseteq \Pi$. The nonempty fibers of $\alpha_1 : R^{(1,1)} \to R^{(1,0)}$ are affine spaces modeled on $(\mathcal{A}^{k+1}_R)_{U_{n-1}}$ where $(\mathcal{A}^{k+1}_R)_{U_{n-1}}$ denotes the subbundle of $\mathcal{A}^{k+1}_R$ consisting of those polynomials which vanish under contraction by vectors in the universal bundle U_{n-1}. In particular, observe the nonempty fibers of α_1 have dimension s_n. On the other hand, the fibers of $\beta_1 : R^{(1,1)} \to R^{(1,0)}$ are dense open subsets of the fibers of the Grassmannian of hyperplanes of U, and so they have dimension $n-1$.

The following lemma will be the basis for our inductive approach to solving the initial value problem.

Lemma 4.4.1. *Let $R \subseteq J^k_n(M)$ be a k-th order differential equation, integrable to first order. If $\alpha_1 : R^{(1,1)} \to R^{(1,0)}$ is surjective then $\alpha_1 : R^{(1)(1,1)} \to R^{(1)(1,0)}$ is also surjective.*

Proof. Let Π_{n-1} be an element in the fiber of $R^{(1)(1,0)}$ at $y \in R^{(1)}$. Let X be the projection of Π_{n-1} to U_y and let $Q_X = U_y/X$. Extend Π_{n-1} to an n-dimensional subspace $\Pi \subseteq (\mathcal{C}_n^{k+1}|_{R^{(1)}})_y$ complementing the vertical $V\pi_{k+1,k}|_{R^{(1)}}$.

Since the form $\overline{[\cdot,\cdot]}$ vanishes when restricted to Π_{n-1}, we have that $C(\Pi)$ belongs to $\mathcal{A}^k_{R,y} \otimes X^* \otimes Q^*_X$. Consider the following sequence

$$\mathcal{A}^{k+1}_{R,y} \otimes U^*_y \xrightarrow{\delta^1} \mathcal{A}^k_{R,y} \otimes \Lambda^2 U^*_y \xrightarrow{\delta^2} (S^{k-1}U^*_y \otimes Q_y) \otimes \Lambda^3 U^*_y \tag{49}$$

which computes the cohomology $H^{2,k+2}(\mathcal{A}_{R,y})$. If we only consider forms which vanish when restricted to X, we get a sequence

$$\mathcal{A}^{k+1}_{R,y} \otimes Q^*_X \xrightarrow{\delta^1} \mathcal{A}^k_{R,y} \otimes X^* \otimes Q^*_X \xrightarrow{\delta^2} (S^{k-1}U^*_y \otimes Q_y) \otimes \Lambda^2 X^* \otimes Q^*_X \tag{50}$$

We claim that the cohomology of this sequence vanishes. This implies that one may find $\Delta \in \mathcal{A}^{k+1}_{R,y} \otimes Q^*_X$ such that $C(\Pi + \Delta) = 0$. The sesqui-holonomic jet $\Pi + \Delta$ is then an element in the fiber of $R^{(1)(1,1)}$ over $R^{(1)(1,0)}$.

Let K be the kernel of the first Spencer coboundary map

$$\mathcal{A}^k_{R,y} \otimes X^* \to (S^{k-1}U^*_y \otimes Q_y) \otimes \Lambda^2 X^* \tag{51}$$

We claim that

$$\dim K = \dim \mathcal{A}^{k+1}_{R,y} - s_n(y) \tag{52}$$

from which the exactness of (50) would follow, since the kernel of the map δ^1 in (50) is $(\mathcal{A}^{k+1}_{R,y})_X$, and the kernel of δ^2 is $K \otimes Q^*_X$.

To see that, let $R^{(1,0)}_X$ be the subset of the fiber of $R^{(1,0)}$ over $y_k = \pi_{k+1,k}y$ given by those planes E_{n-1} such that the projection of E_{n-1} to U_y equals X. By proposition 4.1.1, this is an affine space modeled on K. Consider the map $R^{(1)}_{y_k} \to R^{(1,0)}_X$ which sends each element z in the fiber of $R^{(1)}$ over y_k to the lift of X to $U^{(k)}_z$. This is an affine bundle modeled on the trivial vector bundle with fiber $(\mathcal{A}^{k+1}_R)_X$, which has dimension $s_n(y)$ (here we use that $R^{(1,1)} \to R^{(1,0)}$ is surjective). Putting this all together, we have

$$\dim R^{(1)}_{y_k} = \dim K + s_n(y) \tag{53}$$

from which (52) follows. □

The following lemma guarantees that $R^{(1,0)}$ will be smooth under the conditions of 4.4.1, provided that s_n is constant.

LEMMA 4.4.2. *Let $R \subseteq J^k_n(M)$ be a k-th order differential equation, integrable to first order and with constant s_n. Suppose that $\alpha_1 : R^{(1,1)} \to R^{(1,0)}$ is surjective. Then $R^{(1,0)}$ is smooth and $\pi_{1,0} : R^{(1,0)} \to M$ is a surjective submersion.*

Proof. Let $G^g_{n-1}(U)$ be the bundle over R whose fiber over $y \in R$ consists of those hyperplanes of U_y generic for $\mathcal{A}^k_{R,y}$. The nonempty fibers of the canonical projection

$$R^{(1,0)} \to G^g_{n-1}(U) \tag{54}$$

are affine spaces modeled on the fibers of $\mathcal{A}^k_R|^{(1)}_{U_{n-1}}$, where U_{n-1} denotes the universal bundle on $G^g_{n-1}(U)$, and $\mathcal{A}^k_R|^{(1)}_{U_{n-1}}$ is the first prolongation of the first order tableau bundle $\mathcal{A}^k_R|_{U_{n-1}} \subseteq U^*_{n-1} \otimes (S^{k-1}U^* \otimes Q)$. From (52) in the previous lemma, we see that $\mathcal{A}^k_R|^{(1)}_{U_{n-1}}$ is a smooth bundle over $G^g_{n-1}(U)$. The fact that R is integrable to first order implies that (54) is surjective, since any plane in $G^g_{n-1}(U)$ may be lifted to a hyperplane of an integral element of the contact system. This may be done smoothly, so (54) is a smooth affine bundle and the lemma follows. □

4.5 Let R be a smooth k-th order equation. For each $l \geq 0$, set

$$R^{(l,0)} = J^k_n(M)^{(l,0)} \cap J^l_{n-1}(R) \cap J^{l-1}_{n-1}(R^{(1,0)}) \tag{55}$$

If $R^{(1,0)}$ is smooth, then $R^{(l,0)}$ is its $(l-1)$-th prolongation. Let $R^{(l,l)} = J^k_n(M)^{(l,l)} \cap (R^{(l,0)} \times R^{(l)})$.

We may now state our main theorem regarding initial value problems

THEOREM 4.5.1. *Let $R \subseteq J^k_n(M)$ be a smooth k-th order differential equation, j-regular for $j \geq k+1$ and with constant s_n. Assume that R is integrable to first order and that $R^{(1,1)} \to R^{(1,0)}$ is surjective. Let $l \geq 1$. If $R^{(1,0)}$ is integrable up through order $l-1$, then R is integrable up through order l and we have that the projection $\alpha_l : R^{(l,l)} \to R^{(l,0)}$ is a surjective submersion.*

Proof. We already know that this holds for $l = 1$. Let $l \geq 2$ and assume that the theorem holds for $m < l$.

We first prove that R is integrable to second order. Let $\Pi \in R^{(1)}$, and let Π_{n-1} be a generic hyperplane of Π. Set $y = \pi_{k+1,k}\Pi$. We know that Π_{n-1} may be extended to an $(n-1)$-dimensional submanifold $i_{n-1} : N_{n-1} \to R$ such that the 2-jet of N_{n-1} at y belongs to $R^{(2,0)}$. Let $s : N_{n-1} \to R^{(1)}$ be a section such that the induced section

$$N^{(1)}_{n-1} \to J^1_{n-1}(R) \times_R R^{(1)} \tag{56}$$

is tangent to $R^{(1,1)}$ with order 1 at $i^{(1)}_{n-1}(y)$. This implies that the 2-jet of $s(N_{n-1})$ is tangent to $\mathcal{C}^{k+1}_n$ at $s(y) = \Pi$ with order 2. Therefore, $T_\Pi s(N_{n-1})$ belongs to the fiber of $R^{(1)(1,0)}$ over Π. Since Π was arbitrary, we see that $R^{(1)(1,0)} \to R^{(1)}$ is surjective, and by lemma 4.4.1 we have that R is integrable to second order, as we wanted.

We now claim that $R^{(1)(1,0)}$ is integrable to order $l-2$. By the inductive hypothesis, we know that it is integrable to order $l-3$. Let $z \in R^{(1)(l-2,0)}$. Let $y = \pi_{l-2,0}z \in R^{(1)}$

and $y_k = \pi_{k+1,k} y$. Observe that $\pi^{(l-2)}_{k+1,k} z$ belongs to $R^{(l-2,0)}$. Let $w \in R^{(l,0)}$ be a jet such that $\pi_{l,l-2} w = z$. As usual, we represent w by a submanifold N_{n-1}, and choose a section $s : N_{n-1} \to R^{(1)}$ such that the induced section

$$N^{(1)}_{n-1} \to J^1_{n-1}(R) \times_R R^{(1)} \tag{57}$$

is tangent to $R^{(1,1)}$ with order $l-1$ at y. This implies that the l-jet of $s(N_{n-1})$ is tangent to $\mathcal{C}^{k+1}_n$ and therefore the $(l-1)$-jet belongs to $R^{(1)(l-1,0)}$. The section s may be chosen so that the $(l-2)$-jet of $s(N_{n-1})$ at y equals z. Therefore, the fiber of $R^{(1)(l-1,0)} \to R^{(1)(l-2,0)}$ over z is nonempty. Since z was arbitrary, we see that $R^{(1)(l-1,0)} \to R^{(1)(l-2,0)}$ is surjective. Moreover, this construction may be done so as to depend smoothly over z, so we have that $R^{(1)(l-2,0)}$ is integrable to first order, as we claimed.

We now have that $R^{(1)}$ falls into the hypothesis of the theorem for $l-1$, so we have that $R^{(1)}$ is integrable up through order $l-1$, which implies that R is integrable up through order l. Moreover, starting with an element $w \in R^{(l,0)}$ one may construct an element $w' \in R^{(1)(l-1,0)}$ as above. By the inductive hypothesis, this may be extended to a jet $u \in R^{(k+l)}$ containing w'. This implies that u contains w, and since w was arbitrary, we get that $R^{(l,l)} \to R^{(l,0)}$ is surjective. Since w' may be taken to depend smoothly on w, it is also a submersion, which is what we had to prove. □

4.6 Our next goal is to generalize the theory in this section, to deal with the initial value problem when the initial conditions are given along a submanifold of arbitrary codimension.

Let M be a manifold and $n \leq \dim M$. For each $d \leq n$, let $J^k_n(M)_d$ be the set of d-dimensional planes Π_d tangent to the contact distribution on $J^k_n(M)$, transverse to the vertical $V\pi_{k,k-1}$, and such that $\overline{[\cdot,\cdot]}|_{\Pi_d} = 0$. Denote by U_d the universal bundle on $J^k_n(M)_d$. Let $G_d(U)$ be Grassmannian of d-dimensional subspaces of the universal bundle U over $J^k_n(M)$. Generalizing proposition 4.1.1, we have

Proposition 4.6.1. *The projection $J^k_n(M)_d \to G_d(U)$ is a smooth affine bundle modeled on the kernel of the first Spencer coboundary map*

$$(S^k U^* \otimes Q) \otimes U^*_d \to (S^{k-1} U^* \otimes Q) \otimes \Lambda^2 U^*_d \tag{58}$$

Moreover, the arguments in propositions 4.1.2 and 4.1.3 generalize to yield proofs of the following propositions

Proposition 4.6.2. *Every plane in $J^k_n(M)_{d-1}$ extends to a plane in $J^k_n(M)_d$.*

Proposition 4.6.3. *The first order equation $J^k_n(M)_d$ is formally integrable.*

4.7 Let $R \subseteq J^k_n(M)$ be a smooth k-th order differential equation. Let $R_d \subseteq J^k_n(M)_d$ be the set of generic d-dimensional planes Π_d tangent to R such that $\overline{[\cdot,\cdot]}|_{\Pi_d} = 0$. Let $A_{R,d}$ be the kernel of the following Spencer coboundary map

$$\mathcal{A}^k_R \otimes U^*_d \xrightarrow{\delta^1} (S^{k-1}U^* \otimes Q) \otimes \Lambda^2 U^*_d \tag{59}$$

Observe that the nonempty fibers of the projection $R_d \to G_d(U)$ are affine subspaces of the fibers of $J^k_n(M)_d \to G_d(U)$, modeled on the fibers of $A_{R,d}$. Let $G^g_d(U)$ be the subbundle of $G_d(U)$ over R consisting of the generic d-dimensional subspaces of U. If R_d is smooth and the projection $R_d \to G^g_d(U)$ is a surjective submersion, we have an exact sequence

$$0 \to A_{R,d} \to \mathcal{A}^1_{R_d} \to (U/U_d) \otimes U^*_d \to 0 \tag{60}$$

Lemma 4.7.1. *Let $R \subseteq J^k_n(M)$ be a smooth k-th order differential equation. Let $1 \leq d \leq n$. Suppose that R_d is smooth, the projection $R_d \to G^g_d(U)$ is a surjective submersion, and moreover every plane in R_{d-1} extends to a plane in R_d. Then the projection $R^{(1,1)}_d \to R^{(1,0)}_d$ is surjective.*

Proof. The proof of this will be similar to the one given for lemma 4.4.1. However, it requires some extra steps, and moreover this lemma is fundamental to the theory, so we shall give the full argument in detail.

Let Π_{d-1} be an element in the fiber of $R^{(1,0)}_d$ over $z \in R_d$. Let $y = \pi_{1,0}z \in R$. Let U_d be the universal bundle on R_d. Let X_{d-1} and X_d be the projections of Π_{d-1} and $U_{d,z}$ to U_y, and set $Q_X = X_d/X_{d-1}$. Extend Π_{d-1} to a sesqui-holonomic jet $\Pi_d \subseteq T_z R_d$. Observe that $C(\Pi_d)$ vanishes when restricted to X_{d-1}, so we have

$$C(\Pi_d) \in (T_y R/U_{d,z}) \otimes X^*_{d-1} \otimes Q^*_d \tag{61}$$

By the previous lemma, we know that there exists an integral element of the contact system $\bar{\Pi}_d \subseteq T_z J^k_n(M)_d$. Hence, we have

$$C(\Pi_d) = \delta^1(\bar{\Pi}_d - \Pi_d) \in (\mathcal{C}^k_{n,y}/U_{d,z}) \otimes \Lambda^2 X^*_d \tag{62}$$

Putting (61) and (62) together, we get

$$C(\Pi_d) \in (\mathcal{C}^k_n|_{R,y}/U_{d,z}) \otimes X^*_{d-1} \otimes Q^*_d \tag{63}$$

Denote by P_d this space. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{R,d} \otimes Q^*_d & \longrightarrow & \mathcal{A}^1_{R_d,z} \otimes Q^*_d & \longrightarrow & U_y/X_d \otimes X^*_d \otimes Q^*_d & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \delta^1} & & \big\downarrow{\scriptstyle \delta^1} & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{A}^k_{R,y} \otimes X^*_{d-1} \otimes Q^*_d & \longrightarrow & P_d & \longrightarrow & U_y/X_d \otimes X^*_{d-1} \otimes Q^*_d & \longrightarrow & 0
\end{array} \tag{64}$$

The right vertical arrow is surjective. Therefore, changing Π_d if necessary, one may assume that $C(\Pi_d)$ belongs to $\mathcal{A}^k_{R,y} \otimes X^*_{d-1} \otimes Q_d$.

This space sits inside a sequence

$$A_{R,d} \otimes Q^*_d \xrightarrow{\delta^1} \mathcal{A}^k_{R,y} \otimes X^*_{d-1} \otimes Q^*_d \xrightarrow{\delta^2} (S^{k-1}U^* \otimes Q)_y \otimes \Lambda^2 X^*_{d-1} \otimes Q^*_d \tag{65}$$

We claim that $C(\Pi_d)$ is δ^2-closed. To see this, recall that one has the identity (62) for any $\bar{\Pi}_d$ integral element of the contact system of $J^k_n(M)_d$ at z. Choose $\Pi \subseteq T_y J^k_n(M)$ an integral element of the contact system, so that the sequence

$$0 \to A_{J^k_n(M),d} \to \mathcal{A}^1_{J^k_n(M)_d} \to (U/U_d) \otimes U^*_d \to 0 \tag{66}$$

splits at y. Therefore, one may write

$$\bar{\Pi}_d - \Pi_d = \Delta_1 + \Delta_2 \tag{67}$$

with $\Delta_1 \in (A_{J^k_n(M),d})_y \otimes X^*_d$ and $\Delta_2 \in U_y/X_d \otimes X^*_d \otimes X^*_d$. We have

$$C(\Pi) = \delta^1(\bar{\Pi}_d - \Pi_d) = \delta^1(\Delta_1) + \delta^1(\Delta_2) \tag{68}$$

The fact that $C(\Pi)$ belongs to $\mathcal{A}^k_{R,y} \otimes X^*_{d-1} \otimes Q^*_d$ (and, in particular, to $(S^k U^* \otimes Q)_y \otimes \Lambda^2 X^*_d$) implies that $\delta^1(\Delta_2) = 0$. Therefore, $C(\Pi)$ belongs to the image of

$$(A_{J^k_n(M),d})_y \otimes X^*_d \xrightarrow{\delta^1} (S^k U^* \otimes Q)_y \otimes \Lambda^2 X^*_d \tag{69}$$

Now, observe that

$$(A_{J^k_n(M),d})_y \oplus (S^{\leq k} U^* \otimes Q)_y \tag{70}$$

has a structure of SX^*_d comodule, and the map (69) is part of the complex which computes its Spencer cohomology. The second coboundary of this complex is the usual Spencer coboundary

$$(S^k U^* \otimes Q)_y \otimes \Lambda^2 X^*_d \xrightarrow{\delta^2} (S^{k-1} U^* \otimes Q)_y \otimes \Lambda^3 X^*_d \tag{71}$$

Since $C(\Pi)$ is in the image of (69), we have $\delta^2 C(\Pi) = 0$, as we claimed.

The only thing that remains it to prove that (65) is exact. This is equivalent to the equality

$$\dim (A_{R,d-1})_y = \dim (A_{R,d})_y - \dim ((A_{R,d})_y)_{X_{d-1}} \tag{72}$$

where $((A_{R,d})_y)_{X_{d-1}}$ denotes the subspace of $(A_{R,d})_y$ consisting of those elements which vanish under contraction by all vectors in X_{d-1}.

Denote by $R_{d-1,X_{d-1}}$ (resp. R_{d,X_d}) the subset of R_{d-1} (resp. R_d) consisting of those planes whose projection to U equals X_{d-1} (resp. X_d). Recall that $R_{d-1,X_{d-1}}$ is an affine space modeled on $(A_{R,d-1})_y$ and R_{d,X_d} is an affine space modeled on $(A_{R,d})_y$. Moreover, we have an affine bundle

$$R_{d,X_d} \to R_{d-1,X_{d-1}} \tag{73}$$

modeled on the trivial vector bundle with fiber $((A_{R,d})_y)_{X_{d-1}}$. Therefore, we have

$$\dim A_{R,d} = \dim A_{R,d-1} + \dim((A_{R,d})_y)_{X_{d-1}} \tag{74}$$

as we wanted. □

4.8 Following the same strategy as in the case $d = n$, one may now prove

THEOREM 4.8.1. *Let $R \subseteq J^k_n(M)$ be a smooth k-th order differential equation. Let $d \leq n$ and assume that R_d is j-regular for all j and that s_d is constant. Suppose that $R_d \to G^g_d(U)$ is a surjective submersion and that every plane $\Pi_{d-1} \in R_{d-1}$ extends to a plane $\Pi_d \in R_d$. Let $l \geq 0$. If R_{d-1} is integrable to order l then R_d is integrable to order l. Moreover, every jet in $R^{(l)}_{d-1}$ may be extended (smoothly) to a jet in $R^{(l)}_d$.*

This is related to the Cartan-Kahler theorem in the theory of analytic exterior differential systems (see [1]).

One possible strategy for proving that a differential equation is formally integrable is to try to apply theorem 4.8.1 inductively for R_d, $1 \leq d \leq n$. It turns out that the class of equations for which this is possible are the involutive equations.

THEOREM 4.8.2. *Let $R \subseteq J^k_n(M)$ be a smooth k-th order differential equation with constant Cartan characters. The following are equivalent*

1. *R is involutive.*
2. *For every plane $\Pi_{d-1} \in R_{d-1}$ (with $1 \leq d \leq n$) there exists $\Pi_d \in R_d$ with $\Pi_{d-1} \subseteq \Pi_d$.*

Proof. First, assume that 2 holds. Observe that one may construct points in $R_n = R^{(1)}$ by induction, starting with elements in $R_0 = R$, and so we have that $\pi_{k+1,k} : R^{(1)} \to R$ is surjective. Now, let $y \in R$ and choose a generic flag $X_0 \subseteq X_1 \subseteq \ldots X_n$ for U_y. Let $(R_d)_{X_d}$ be the submanifold of $R_{d,y}$ consisting of those planes Π_d whose projection to U_y equals X_d. Observe that we have a chain of projections

$$R^{(1)}_y = R_{n,y} \to (R_{n-1})_{X_{n-1}} \to (R_{n-2})_{X_{n-2}} \to \ldots \to (R_1)_{X_1} \to \{y\} \tag{75}$$

where the map $(R_d)_{X_d} \to (R_{d-1})_{X_{d-1}}$ sends Π_d to the lift of X_{d-1} to Π_d. Each $(R_d)_{X_d}$ is an affine space and the projection $(R_d)_{X_d} \to (R_{d-1})_{X_{d-1}}$ is an affine bundle modeled on the trivial vector bundle with fiber $((A_{R,d})_y)_{X_{d-1}}$. Therefore, one has

$$\dim \mathcal{A}^{k+1}_{R,y} = \sum_{d=1}^{n} \dim((A_{R,d})_y)_{X_{d-1}} \tag{76}$$

Now, the space $((A_{R,d})_y)_{X_{d-1}}$ may be computed as the kernel of

$$\mathcal{A}^k_{R,y} \otimes (X_d/X_{d-1})^* \xrightarrow{\delta^1} (S^{k-1}U^* \otimes Q)_y \otimes X^*_{d-1} \otimes (X_d/X_{d-1})^* \tag{77}$$

and so we have $((A_{R,d})_y)_{X_{d-1}} = (\mathcal{A}^k_{R,y})_{X_{d-1}} \otimes (X_d/X_{d-1})^*$. The equality (76) becomes

$$\dim \mathcal{A}^{k+1}_{R,y} = \sum_{d=1}^{n} \dim(\mathcal{A}^k_{R,y})_{X_{d-1}} \tag{78}$$

and so $\mathcal{A}^k_{R,y}$ satisfies Cartan's test for involutivity.

Conversely, if R is involutive, for each $y \in R$ and generic flag $X_0 \subseteq X_n$ for U_y we have that (76) holds. Observe that each map in (75) is a morphism of affine spaces. From (76) we get that all the maps in (75) are surjective, and the theorem follows. □

As we commented before, the formal integrability of involutive equations actually follows from 3.3.2, so this inductive approach is not needed in order to find solutions. Nevertheless, there are some advantages to knowing that a formally integrable equation is involutive. For instance, one may estimate the size of the space of solutions to involutive equations, using that such estimates are possible for each initial value problem. Moreover, the construction of Spencer complexes in the linear theory (essentially, resolutions of the sheaf of solutions to formally integrable linear differential equations) becomes specially simple in the involutive case. We refer the reader to [1] for a full discussion of involutivity and the applications mentioned above.

Chapter IV

Equivalence Problems

In this chapter we study Cartan's method for obtaining a complete set of invariants of a geometric structure. This is also called the equivalence method, since it may be used to decide when two structures are locally isomorphic (at least in the analytic case).

In section 1 we introduce the concept of a G-structure on a manifold M. This is a reduction of the structure group of the principal $GL_n(\mathbb{R})$ bundle of frames FM to a subgroup $G \subseteq GL_n(\mathbb{R})$. Several classic geometric structures may be interpreted in this way. For example, $O_n(\mathbb{R})$-structures correspond to Riemannian metrics, and in the $2n$-dimensional case $GL_n(\mathbb{C})$-structures correspond to almost complex structures. We define the essential torsion of a G-structure, which is the obstruction for the structure to being flat to second order. Equivalently, this is the obstruction to the existence of torsion free connections for the structure.

In section 2 we begin the study of Cartan's method. We first show how this works in the case when G is the trivial group, in which case the method gives a complete set of invariants for a coframing on a manifold (i.e., a trivialization of the cotangent bundle). We then present the general case, which consists of three steps: normalizing the invariants to reduce the structure group, checking if the conditions of our formal integrability theorem hold, and prolongation.

Section 3 is an introduction to the theory of semi-holonomic jets. In the same way that k-jets of submanifolds of a manifold M correspond in coordinates to polynomials of degree k in commuting variables, semi-holonomic jets correspond to polynomials of degree k in non-commuting variables. Most of the definitions and results in this section have a holonomic analogue found in the earlier chapters of this thesis.

In section 4 we discuss the theory of semi-holonomic higher order G-structures. These are needed in order to understand the process of prolongation in the equivalence method. We define the total curvature of a higher order G-structure and prove our main equivalence result which shows that, if a structure has been normalized, the derivatives of its total curvature completely characterize it (at a formal level), provided that certain Spencer cohomology groups vanish. We finish by giving a complete description of the equivalence method and prove that it terminates in a finite number of steps.

1 G-Structures

1.1 Let M be an n-dimensional manifold. Let $J^k_{iso}(M,\mathbb{R}^n)$ be the open subset of $J^k_n(M\times\mathbb{R}^n)$ consisting of those k-jets which are transverse to the horizontal and vertical distributions TM and $T\mathbb{R}^n$ on $M\times\mathbb{R}^n$. In other words, $J^k_{iso}(M,\mathbb{R}^n)$ consists of the k-jets of diffeomorphisms between M and $\mathbb{R}^n$.

We define the *k-th order frame bundle* of M as $F^k(M)=\pi^{-1}_{k,0}(M\times\{0\})$. In particular, $F(M)=F^1(M)$ is the bundle of frames of M (where a frame at $x\in M$ consists of a basis for the tangent space T_xM). Observe that $J^k_{iso}(M,\mathbb{R}^n)=F^k(M)\times\mathbb{R}^n$.

We have an isomorphism $TM=T\mathbb{R}^n$ of bundles over $J^1_{iso}(M,\mathbb{R}^n)$. Let $\theta^1,\ldots,\theta^n$ be the induced basis for T^*M. These are called the *tautological (or canonical) forms.* Observe that the forms $\theta^i-\mathrm{d}x^i$ give a basis for the contact forms on $J^1_{iso}(M,\mathbb{R}^n)$. Moreover, the contact distribution on $J^1_{iso}(M,\mathbb{R}^n)=F(M)\times\mathbb{R}^n$ may be identified with $TF(M)$. An n-dimensional plane $\Pi\subseteq T_\omega F(M)$ induces an integral element of the contact system at $(\omega,x)\in F(M)\times\mathbb{R}^n=J^1_{iso}(M,\mathbb{R}^n)$ if and only if it is transverse to $V_\omega\pi_{1,0}$ and $\mathrm{d}\theta^i|_\Pi$ vanishes for all i. Therefore, $F^2(M)$ defines a first order equation on sections of $F(M)\to M$. Solutions to this equation are coframings ω^i on M such that $\mathrm{d}\omega^i=0$ for all i. Locally, Poincaré's lemma implies that these define a coordinate system x^i on M such that $\mathrm{d}x^i=\theta^i$.

Using the linearization theorem I.2.6.3 in the case of the bundle $\xi: M\times\mathbb{R}^n\to M$, we have that $V(\xi^{(k)})=J^k_n(\mathbb{R}\to M)$. When working over $J^k_{iso}(M,\mathbb{R}^n)$, this may be identified with $J^k_n(\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n)=\bigoplus_{0\le j\le k}S^j\mathbb{R}^{n*}\otimes\mathbb{R}^n$. Let ψ^a_I be the induced basis on $V\pi^*_{k,0}$, where $1\le a\le n$ and I is a symmetric multi-index of length $1\le|I|\le k$. When working over $J^{k+1}_{iso}(M,\mathbb{R}^n)$, the universal bundle complements the vertical distribution $V\pi_{k,0}$, and so we have a trivialization of $T^*J^k_{iso}(M,\mathbb{R}^n)$ given by the forms $\theta^i,\mathrm{d}x^a,\psi^a_I$. Observe that the forms θ^i and ψ^a_I annihilate the distribution $T\mathbb{R}^n$, and so they may be thought of as forms on F^kM. These give a trivialization of T^*F^kM over $F^{k+1}M$, and are called the *canonical forms* on $F^{k+1}M$. From this, we get an embedding $F^{k+1}M\subseteq FF^kM$.

The contact bundle on $J^{k+1}_{iso}(M,\mathbb{R}^n)$ is spanned by the forms $\theta^i-\mathrm{d}x^i,\psi^a_I$. The projection $J^{k+1}_{iso}(M,\mathbb{R}^n)\to F^{k+1}M$ induces an isomorphism between the contact distribution and the bundle $TF^{k+1}M$. An n-dimensional subspace $\Pi\subseteq T_\omega F^{k+1}M$ complementing $V\pi_{k+1,k}$ corresponds to an integral element of the contact system on $J^{k+1}_{iso}(M,\mathbb{R}^n)$ if and only if the forms $\mathrm{d}\theta^i$ and $\mathrm{d}\psi^a_I$ vanish when restricted to Π. Therefore, $F^{k+2}M$ may be considered as a first order equation on sections of $F^{k+1}M\to F^kM$. The contact system for this equation is spanned by the forms θ^i,ψ^a_I (where $1\le|I|\le k+1$). From this, one may see, inductively, that $F^{k+2}M$ is the k-th prolongation of the first order equation F^2M on sections of $FM\to M$.

1.2 Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be a Lie subgroup. A *G-structure* on M is a reduction F_G of the structure group of the $GL_n(\mathbb{R})$ principal bundle $F(M)$ to G. In other words, this is a subfibered manifold $F_G \subseteq F(M)$ such that the right $GL_n(\mathbb{R})$ action on $F(M)$ restricts to an action of G on F_G giving F_G the structure of a principal G-bundle.

The submanifold $F_G \times \mathbb{R}^n \subseteq F(M) \times \mathbb{R}^n = J^1_{\mathrm{iso}}(M, \mathbb{R}^n)$ is a first order equation. Its symbol is the vertical distribution on F_G, which is a trivial bundle with fiber $\mathfrak{g}$ the Lie algebra of G. For each $\omega \in F_G$, the curvature of the equation at a point (ω, x) does not depend on x, and belongs to the space $H^{2,2}(\mathfrak{g})$. This gives a section of the trivial bundle with fiber $H^{2,2}(\mathfrak{g})$ over F_G called the *essential torsion* of the G-structure.

The prolongations of $F_G \times \mathbb{R}^n$ have the form $F^k_G \times \mathbb{R}^n \subseteq F^k(M) \times \mathbb{R}^n$. The G-equivariant sections of $F^2_G \to F_G$ are in correspondence with distributions $\mathcal{H}$ on F_G transverse to the vertical, such that the two-forms $\mathrm{d}\theta^i$ vanish along $\mathcal{H}$. These are the torsion free principal connections on F_G. Therefore, the vanishing of the essential torsion of the G-structure is equivalent to the existence of a torsion-free connection.

Set $\mathfrak{g}^{(1)} = (\mathbb{R}^{n*} \otimes \mathfrak{g}) \cap (S^2\mathbb{R}^{n*} \otimes \mathbb{R}^n)$. If the essential torsion vanishes then $F^2_G \to F_G$ is an affine bundle modeled on the trivial bundle with fiber $\mathfrak{g}^{(1)}$. Therefore, principal connections are an affine bundle modeled on the vector space of G-equivariant sections of $F_G \times \mathfrak{g}^{(1)}$, which is the same thing as the space of sections of the bundle associated to F_G with fiber $\mathfrak{g}^{(1)}$.

We may interpret a torsion free principal connection as giving a (G-equivariant) way of extending frames in F_G to 2-jets of coordinate systems in F^2_G. In the presence of such a connection, one has a distinguished class of 2-jets of coordinate systems on M.

Fix a torsion free principal connection ∇ on F_G. Let $s : F_G \to F^2_G$ be the associated section. Consider the second order differential equation $R_\nabla = \mathrm{im}(s) \times \mathbb{R}^n \subseteq J^2_{\mathrm{iso}}(M, \mathbb{R}^n)$. Solutions of this equation are in correspondence with local (connection preserving) isomorphisms between (M, F_G) and $\mathbb{R}^n$ with the canonical (flat) G-structure and connection.

Observe that we have $\mathcal{A}^0_{R_\nabla} = \mathbb{R}^n$, $\mathcal{A}^1_{R_\nabla} = \mathfrak{g}$ and $\mathcal{A}^j_{R_\nabla} = 0$ for $j \geq 2$. The curvature of R_∇ is then a G-equivariant function from $R_\nabla = F_G \times \mathbb{R}^n$ to $H^{2,2}(\mathfrak{g}) = \ker(\delta^2 : \mathfrak{g} \otimes \Lambda^2\mathbb{R}^{n*} \to \mathbb{R}^n \otimes \Lambda^3\mathbb{R}^{n*})$. This is invariant under $\mathbb{R}^n$, and may be identified with the curvature of the connection ∇. The fact that δ^2 annihilates the curvature of ∇ is called the second Bianchi identity.

We may also consider R_∇ as a first order differential equation on submanifolds of $F_G \times \mathbb{R}^n$. Then R_∇ is induced by a distribution as in III.3.4. By the Frobenius theorem, if the curvature vanishes then (F_G, ∇) is locally isomorphic to $\mathbb{R}^n$.

EXAMPLE 1.2.1. Consider the case when G is the orthogonal group $O_n(\mathbb{R})$. An $O_n(\mathbb{R})$-structure $F_{O_n(\mathbb{R})}$ on an n-dimensional manifold M is the same as a Riemannian metric on M, so that $F_{O_n(\mathbb{R})}$ is the bundle of orthonormal frames for the induced metric. Example II.2.4.2 shows that $\mathfrak{o}_n^{(1)}$ and $H^{2,2}(\mathfrak{o}_n)$ vanish. This implies that there exists a unique torsion free connection (indeed, the usual proof of the existence and uniqueness of the Levi-Civita connection depends on the same arguments that we used to show that those two spaces vanish).

The Levi-Civita connection gives us a distinguished class of 2-jets of coordinate systems around each point $q \in M$. The property that these coordinate systems have is that the metric is Euclidean to the first order. That is, if we let $x^1, \ldots, x^n$ be one such system, then the metric g at q is the Euclidean metric $(\mathrm{d}x^1)^2 + \ldots + (\mathrm{d}x^n)^2$, and the directional derivatives $\frac{\partial g}{\partial x^i}$ vanish at q. This is called a normal coordinate system, and is usually constructed using the exponential map (which, of course, depends on the existence of a connection compatible with the metric).

The curvature of the Levi-Civita connection is the obstruction to the existence of 3-jets of coordinate systems where the metric is Euclidean to the second order. This is a manifestation of the fact that the curvature coefficients arise in the second order Taylor expansion of the metric in a normal coordinate system.

2 Cartan's Method

2.1 Let F_G and F'_G be G-structures on two n-dimensional manifolds M and M'. The equivalence problem asks if there exists a local isomorphism between (M, F_G) and (M, F'_G). We already have already solved a particular case of this: a $O_n(\mathbb{R})$-structure is locally isomorphic to $\mathbb{R}^n$ if and only if the curvature vanishes.

A basic case of this is when G is the trivial group e. In this case, a e-structure on M is a coframe $\omega^1, \ldots \omega^n$ (i.e., a global trivialization of T^*M). Given $(M, \omega), (M', \omega')$ two e-structures, the equivalence problem asks if there exists a (locally defined) diffeomorphism $\psi : M \to M'$ such that $\psi^*\omega'^i = \omega^i$ for all i.

Write $\mathrm{d}\omega^i = \lambda^i_{jk}\omega^j \wedge \omega^k$. The numbers λ^i_{jk} are called the *structure coefficients* of the coframe ω. They define a smooth function $\lambda : M \to \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*}$. More generally, for each $K \geq 0$ one may consider the derivatives $\frac{\partial^{|I|}}{\partial w^I}(\lambda)$ of λ for $|I| \leq K$. This may be considered as a function $D^K\lambda : M \to S^{\leq K}\mathbb{R}^{n*} \otimes (\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*})$. We say that λ *stabilizes at order* K around $q \in M$ if $D^K\lambda$ and $D^{K+1}\lambda$ have constant rank near q, and both ranks are the same. It is easily seen that λ stabilizes at order n around q for q in an open dense subset of M. If λ stabilizes at order K around q, then we may write $D^{K+1}\lambda = fD^K\lambda$ for some (locally defined) section $f : S^{\leq K}\mathbb{R}^{n*} \otimes (\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*}) \to S^{\leq K+1}\mathbb{R}^{n*} \otimes (\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*})$. By the chain rule, the same thing is true for higher order derivatives. Therefore, λ also stabilizes at order L for all $L \geq K$.

Let λ' be the structure coefficients of ω'. Observe that in order for there to exist an equivalence sending $q \in M$ to $q' \in M'$, we must have $D^K\lambda(q) = D^K\lambda'(q')$ for all K. Moreover, if λ and λ' stabilize at order K near q and q', then necessarily the ranks of $D^K\lambda$ and $D^K\lambda'$ are the same, and the images of $D^{K+1}\lambda$ and $D^{K+1}\lambda'$ coincide near $D^{K+1}\lambda(q)$. Conversely, we have the following

Proposition 2.1.1. *Let M, M' be two n-dimensional manifolds, and let ω and ω' be coframes for M and M' respectively. Let $q \in M$ and $q' \in M'$. Suppose that λ and λ' stabilize at order K near q and q'. If $D^{K+1}\lambda(q) = D^{K+1}\lambda'(q')$ and the images of $D^{K+1}\lambda$ and $D^{K+1}\lambda'$ coincide near $D^{K+1}\lambda(q)$, then there exists a unique local equivalence sending q to q'.*

Proof. Consider the distribution $\mathcal{C}$ on $M \times M'$ defined by the 1-forms $\omega^i - \omega'^i$ for $1 \leq i \leq n$. Local isomorphisms between (M, ω) and (M', ω') are in correspondence with n-dimensional integral submanifolds of $\mathcal{C}$.

Let $S \subseteq M \times M'$ be the subset consisting of the pairs (y, y') such that $D^{K+1}\lambda(y) = D^{K+1}\lambda'(y')$. The n-dimensional integral submanifolds of $\mathcal{C}$ must necessarily be contained inside S. Near (q, q'), we have that S is a smooth submanifold of codimension r and the projections $S \to M$ and $S \to M'$ are submersions.

Now, write $D^{K+1}\lambda = fD^K\lambda$ and $D^{K+1}\lambda' = f'D^K\lambda'$ near (q, q'). Since the images of $D^{K+1}\lambda$ and $D^{K+1}\lambda'$ coincide, we may take $f = f'$. By the chain rule, we have $D^{K+2}\lambda = gD^K\lambda$ and $D^{K+2}\lambda' = gD^K\lambda'$ for the same function g. Therefore, we conclude that $D^{K+2}\lambda(y) = D^{K+2}\lambda(y')$ for $(y, y') \in S$ near (q, q').

Observe that the distribution $\mathcal{C}$ is spanned by the vector fields $X_i = \partial/\partial\omega^i - \partial/\partial\omega'^i$ for $1 \leq i \leq n$. The fact that $D^{K+2}\lambda$ and $D^{K+2}\lambda'$ coincide implies that $X_i(D^{K+1}\lambda - D^{K+1}\lambda') = 0$ for all i. Therefore, the distribution $\mathcal{C}$ is tangent to S. Since $\lambda - \lambda'$ vanishes along S, we have that $\mathcal{C}$ is Frobenius integrable, and the proposition follows. □

Observe that from the proof of the proposition one also gets a local description of the space of equivalences: it may be parameterised as a smooth manifold of dimension $n - r$, where r is the rank of $D^K\lambda$.

2.2 The general equivalence problem may be solved (in principle) using the method of Élie Cartan. This algorithm is a systematic way of finding invariants for a given G-structure. If the invariants for two structures coincide at each step in the algorithm, then they are shown to be formally equivalent. In the analytic case, this implies the existence of local isomorphism.

We shall present the method in three steps. The first one is normalization, where one uses invariants which vary along the fibers of the structure to reduce the structure group as much as possible. The second one is to check for formal integrability of a certain differential equation whose solutions are the local equivalences. The third step is prolongation, where one passes to higher order structures and starts back at step one.

(i) Let (M, F_G) and (M', F'_G) be two G structures. Let $q \in M$ and $q' \in M'$. We shall work in sufficiently small neighborhoods of q and q'. Assume that G is connected, if not one restricts to the connected components of of the structures. Let $T : F_G \to H^{2,2}(\mathfrak{g})$ and $T' : F'_G \to H^{2,2}(\mathfrak{g})$ be the essential torsions of the two structures.

The first step will be to normalize the torsion to reduce the structure groups. A submanifold $S \subseteq H^{2,2}(\mathfrak{g})$ is called a *normalizing submanifold* if $S \cap \mathcal{O}$ has at most one point and is transverse for all the orbits $\mathcal{O}$ for the right G action on $H^{2,2}(\mathfrak{g})$, and moreover the isotropy of the action is constant along S. Assume that there exists a normalizing submanifold S with isotropy H such that S intersects the orbit of $T(\omega)$ for every coframe $\omega \in F_G$. This happens for example if all those orbits coincide, in which case one may take S to be a single point. If there exists an equivalence sending q to q', then the same must be true for the orbits of $T'(\omega')$ for $\omega' \in F'_G$. In this case, $F_H = T^{-1}(S)$ and $F'_H = T'^{-1}(S)$ are reductions of the structure groups of F_G and F'_G to H, and the equivalences between F_G and F'_G coincide with the equivalences between F_H and F'_H. If H is a proper subgroup of G, one starts the algorithm again with the reduced structures. Observe that one has to compute the essential torsions again with the new structures, so it may happen that further normalization of the torsion is possible after reducing the group.

Now, assume that $H = G$. This means that the action of G fixes the torsions T and T', which implies that T and T' are constant along the fibers of F_G and F'_G. Write $dT = \frac{\partial}{\partial \omega^i} T \omega^i$. This defines a function $DT : F_G \to \mathbb{R}^{n*} \otimes H^{2,2}(\mathfrak{g})$, which is equivariant under the action of G. We may now repeat the same thing that we did for T. Assume that there exists a normalizing submanifold $S \subseteq \mathbb{R}^{n*} \otimes H^{2,2}(\mathfrak{g})$ with isotropy H_1, intersecting the orbits of the points in the image of DT. The same must be true for the image of DT' if the structures are to be equivalent. If H_1 is a proper subgroup of G, one may reduce the structure group to H_1, and start the algorithm again.

If $H_1 = G$, the function DT is constant along the fibers of F_G, and we may consider its derivatives, which define a G-equivariant function $D^2T : F_G \to S^2\mathbb{R}^{n*} \otimes H^{2,2}(\mathfrak{g})$. This process goes on, considering at each step derivatives of T of increasing order. If no further reductions are possible, then all the functions $D^K T$ and $D^K T'$ are constant along the fibers of the structures, so they may be thought as functions defined on M.

(ii) We now assume that T stabilizes at order K near q, meaning that $D^K T$ and $D^{K+1} T$ have constant rank near q, and both ranks coincide. We may take $K = n$ for q in an open dense subset of M. If the structures are to be equivalent, then T' must also stabilize at order K around q'. Moreover, we must have that $D^{K+1}T(q) = D^{K+1}T'(q')$ and the images of $D^{K+1}T$ and $D^{K+1}T'$ must coincide near $D^{K+1}T(q)$.

Let $R \subseteq J^1_{iso}(M, M')$ be the first order differential equation consisting of those 1-jets of diffeomorphisms compatible with the G-structures. Solutions of R are in correspondence with (local) equivalences between (M, F_G) and (M', F'_G). Let $S \subseteq M \times M'$ be the submanifold defined by the equation $D^{K+1}T = D^{K+1}T'$. Observe that all the solutions of R have to be contained inside S. Let $R|_S = J^1_n(S) \cap R$.

PROPOSITION 2.2.1. *We have that $R|_S = R \cap \pi^{-1}_{1,0}(S)$ and its curvature vanishes. Moreover, if $H^{2,j}(\mathfrak{g}) = 0$ for $j \geq 3$, the equation $R|_S$ is formally integrable.*

Proof. This goes along the same lines as the proof of proposition 2.1.1. Let ω and ω' be sections of F_G and F'_G. Then there is an induced section ψ of R over $M \times M'$, whose value at a point (y, y') is the isomorphism $\psi_{y,y'} : T_yM \to T_{y'}M'$ such that $\psi^*_{y,y'}\omega' = \omega$. From this, we get a trivialization $R = M \times M' \times G$, where the point (y, y', g) corresponds to the isomorphism $\psi_{y,y'}g$ such that $(\psi_{y,y'}g)^*(\omega' g) = \omega$. The contact system is defined by the vector valued 1-form $\omega - \omega' g$.

Let $z = (y, y', g) \in R \cap \pi^{-1}_{1,0}(S)$. Then z is an n-dimensional plane at (y, y'), spanned by the vectors $\partial/\partial\omega^i - \partial/\partial(\omega' g)^i$. Using that $D^{K+1}T'$ is invariant under the action of g, we have

$$\left(\frac{\partial}{\partial\omega^i} - \frac{\partial}{\partial(\omega' g)^i}\right)(D^{K+1}T - D^{K+1}T') = \frac{\partial}{\partial\omega^i}D^{K+1}T - \frac{\partial}{\partial\omega'^i}D^{K+1}T' \tag{1}$$

And this vanishes along S by the same arguments as in the proof of 2.1.1. This proves that $R|_S = R \cap \pi^{-1}_{1,0}(S)$, which is the first part of the proposition.

Now, let $z = (y, y', g) \in R$ and Π be the subspace of T_zR defined by the equations $\omega_y - \omega'_{y'}g = 0$ and $\mathrm{d}g = 0$. We have that $\mathrm{d}(\omega - \omega' g)|_\Pi = (\mathrm{d}\omega - (\mathrm{d}\omega')g)|_\Pi \in \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*}$. Since the essential torsions $T(y)$ and $T'(y')$ coincide, we have that $\mathrm{d}(\omega - \omega' g)|_\Pi$ projects to zero in $H^{2,2}(\mathfrak{g})$, which means that the curvature of R vanishes at z.

The last part of the proposition follows from the fact that the symbol of $R|_S$ may be identified with $\mathfrak{g}$ after choosing sections ω and ω' of F_G and F'_G. □

(iii) If $H^{2,j}(\mathfrak{g})$ does not vanish for some $j \geq 3$, one fixes a section $\alpha : H^{2,2}(\mathfrak{g}) \to \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*}$. We consider (non necessarily equivariant) connections on F_G with torsion constant along the fibers, equal to $\alpha T : F_G \to \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*}$. If we fix a basis for $\mathfrak{g}$, the connection forms complement the tautological forms to give a coframing on F_G. These coframings define a $\mathfrak{g}^{(1)}$-structure $F_{\mathfrak{g}^{(1)}}$ on F_G, where $\mathfrak{g}^{(1)} = (\mathbb{R}^{n*} \otimes \mathfrak{g}) \cap (S^2\mathbb{R}^{n*} \otimes \mathbb{R}^n)$ is considered as a subgroup of $GL(\mathbb{R}^n \oplus \mathfrak{g})$ via $L \mapsto L + \mathrm{id}$. This is called a *prolongation* of F_G.

Analogously, one constructs a prolongation of F'_G using the same section α. The following result tells us that both equivalence problems are essentially the same.

PROPOSITION 2.2.2. *If $\psi : M \to M'$ is an equivalence, then its lift $\psi^{(1)} : F_G \to F'_G$ is an equivalence of $\mathfrak{g}^{(1)}$-structures. Conversely, each equivalence of $\mathfrak{g}^{(1)}$-structures $\bar\psi : F_G \to F'_G$ is induced by a unique equivalence $\psi : M \to M'$.*

Proof. The first part of the statement is a consequence of the fact that equivalences preserve the torsion of connections. For the second part, observe that an equivalence $\bar\psi : F_G \to F'_G$ of $\mathfrak{g}^{(1)}$-structures must be compatible with the canonical forms, that is, $\psi^*\theta'^i = \theta^i$. The distributions defined by the canonical forms are the vertical distributions of F_G and F'_G. Since we are assuming that G is connected, we have that $\bar\psi$ commutes with the projections $F_G \to M$ and $F'_G \to M'$, and therefore we get an induced map $\psi : M \to M'$.

Let s and s' be sections of F_G and F'_G such that $\bar{\psi}s = s'\psi$. Then $\psi^* s'^* \theta'^i = s^*\theta^i$ for all i. Since $s^*\theta^i$ and $s'^*\theta'^i$ are coframes belonging to F_G and F'_G, this means that ψ is an equivalence, as we wanted. □

Observe that the result as stated holds for global equivalences (or in neighborhoods of q and q'). However, one could also allow local equivalences between $F_{\mathfrak{g}^{(1)}}$ and $F'_{\mathfrak{g}^{(1)}}$ in which case the same correspondence holds, modulo a suitable equivalence relation.

The algorithm now starts again at the first step with the prolonged structures, although some modifications are to be made. Suppose that at some point we normalize the essential torsion of $F_{\mathfrak{g}^{(1)}}$ so that we get an invariant $F_G \to H^{2,2}(\mathfrak{g}^{(1)})$. If this function is not constant along the fibers of F_G, it may be used to reduce the structure group G to a proper subgroup H. In this case one may start the method again, with the reduced structure F_H.

Further prolongation may be needed. One could simply take, once more, connections on the bundle $F_{\mathfrak{g}^{(1)}} \to F_G$, which are simply sections of $J^1(F_{\mathfrak{g}^{(1)}} \to F_G)$ over $F_{\mathfrak{g}^{(1)}}$. However, $F_{\mathfrak{g}^{(1)}} \to F_G$ is not only a $\mathfrak{g}^{(1)}$-structure, but there is also an inclusion $F_{\mathfrak{g}^{(1)}} \subseteq J^1_n(F_G)$ as well. Therefore, it is natural to consider sections of the bundle $\tilde{F}_{\mathfrak{g}^{(1)}} \to F_{\mathfrak{g}^{(1)}}$ whose fiber over a point $y \in F_{\mathfrak{g}^{(1)}}$ consists of the lifts of $U^{(1)}_y \subseteq T_{\pi_{2,1}y}F_G$ to $T_y F_{\mathfrak{g}^{(1)}}$. Points in $\tilde{F}_{\mathfrak{g}^{(1)}}$ may be thought of as coframes on $F_{\mathfrak{g}^{(1)}}$ with values in the vector space $\mathbb{R}^n \times \mathfrak{e}$, where $\mathfrak{e}$ is a certain extension of $\mathfrak{g}$ by the abelian Lie algebra $\mathfrak{g}^{(1)}$ such that the vertical distribution of the projection $F_{\mathfrak{g}^{(1)}} \to M$ is the trivial bundle with fiber $\mathfrak{e}$.

The process of prolongation may be better understood in terms of semi-holonomic higher order G-structures. After discussing this, we shall be in a position to give a complete description of the method and prove that it terminates in a finite number of steps.

3 Semi-Holonomic Jets

In this section we give a brief introduction to the theory of semi-holonomic jets. This is what one gets if one drops the requirement that derivatives should commute when taking jet prolongation. Most results have a holonomic analogue in the theory that we discussed earlier in this thesis, so we shall skip the proofs.

3.1 Let M be a differentiable manifold and fix $n \leq \dim M$. Let U be the universal bundle on $J^1_n(M)$ and $Q = TM/U$. Set $J^{\langle 0\rangle}_n(M) = M$ and $J^{\langle 1\rangle}_n(M) = J^1_n(M)$. The space $J^{\langle 2\rangle}_n(M)$ of *second order semi-holonomic jets* of n-dimensional submanifolds of M is the bundle over $J^{\langle 1\rangle}_n(M)$ whose fiber over a point $y \in J^{\langle 1\rangle}_n(M)$ consists of the lifts of U_y to $T_y J^{\langle 1\rangle}_n(M)$. Of course, the space $J^2_n(M)$ of (holonomic) 2-jets sits inside $J^{\langle 2\rangle}_n(M)$

as the bundle of integral elements of the contact system on $J^1_n(M)$. Let $U^{\langle 1\rangle}$ be the pullback of the universal bundle on $J^1_n(J^{\langle 1\rangle}_n(M))$ to $J^{\langle 2\rangle}_n(M)$.

Inductively, we define the space $J^{\langle k\rangle}_n(M)$ of *k-th order semi-holonomic jets* of *n*-dimensional submanifolds of M as the bundle over $J^{\langle k-1\rangle}_n(M)$ whose fiber over a point y consists of the lifts of $U^{\langle k-2\rangle}_y$ to $T_y J^{\langle k-1\rangle}_n(M)$, where $U^{\langle k-2\rangle}$ is the pullback of the universal bundle on $J^1_n(J^{\langle k-2\rangle}_n(M))$ to $J^{\langle k-1\rangle}_n(M)$. Let $\pi_{k,j} : J^{\langle k\rangle}_n(M) \to J^{\langle j\rangle}_n(M)$ be the canonical projection. Observe that the space $J^k_n(M)$ of holonomic k-th order jets is contained inside $J^{\langle k\rangle}_n(M)$.

Locally, a coordinate system x^i, u^a on M extends to coordinates x^i, u^a_I on $J^{\langle k\rangle}_n(M)$, where I is a multi-index of length at most k. A point defines a holonomic jet if its coordinates u^a_I are symmetric.

PROPOSITION 3.1.1. *Let $k \geq 2$. The bundle $J^{\langle k\rangle}_n(M) \to J^{\langle k-1\rangle}_n(M)$ is an affine bundle modeled on $(U^*)^{\otimes k} \otimes Q$.*

A *k-th order semi-holonomic differential operator* is a smooth map $\varphi : J^{\langle k\rangle}_n(M) \to M'$, where M' is another smooth manifold. As usual, we may form the l-th prolongation $\varphi^{(l)}$ which is a (partially defined) map from $J^{\langle k+l\rangle}_n(M)$ to $J^{\langle l\rangle}_n(M')$. The symbol of φ is the map $\sigma_\varphi : (U^*)^{\otimes k} \otimes Q$ obtained by restriction of φ_* to $V\pi_{k,k-1}$. If $l \geq 2$, we have that $\varphi^{(l)}$ is a affine bundle map over $\varphi^{(l-1)}$ modeled on $\sigma^{(l)}_\varphi = 1_{(U^*)^{\otimes l}} \otimes \sigma_\varphi$. Prolongation of the universal differential operator $\mathrm{id} : J^{\langle k\rangle}_n(M) \to J^{\langle k\rangle}_n(M)$ gives the canonical inclusion $J^{\langle k+l\rangle}_n(M) \hookrightarrow J^{\langle l\rangle}_n(J^{\langle k\rangle}_n(M))$.

All the theory may be extended to the fibered case in the usual way. The following linearization result may be proven inductively

PROPOSITION 3.1.2. *1. There is a short exact sequence of bundles over $J^{\langle k+1\rangle}_n(M)$*

$$0 \to \tilde{\mathcal{H}}^k \to J^{\langle k\rangle}_n(TM \to M) \to TJ^{\langle k\rangle}_n(M) \to 0 \tag{2}$$

where $\tilde{\mathcal{H}}^k$ is the kernel of the canonical map

$$J^{\langle k\rangle}_n(U \to J^1_n(M)) \to U \tag{3}$$

2. There is a short exact sequence of bundles over $J^{\langle k+1\rangle}_n(M)$

$$0 \to U^{\langle k\rangle} \to TJ^{\langle k\rangle}_n(M) \to J^{\langle k\rangle}_n(Q \to J^1_n(M)) \to 0 \tag{4}$$

3.2 We define the contact distribution on $J^{\langle k\rangle}_n(M)$ as $\tilde{\mathcal{C}}^k_n = \pi^{-1}_{k,k-1*}(U^{\langle k-1\rangle})$. As usual, the Lie bracket induces a form

$$\overline{[\cdot,\cdot]} : \Lambda^2\tilde{\mathcal{C}}^k_n \to TJ^{\langle k\rangle}_n(M)/\tilde{\mathcal{C}}^k_n = J^{\langle k-1\rangle}_n(Q \to J^1_n(M)) \tag{5}$$

The induced map

$$((U^*)^{\otimes k} \otimes Q) \otimes U = V\pi_{k,k-1} \otimes U \to J_n^{\langle k-1 \rangle}(Q \to J_n^1(M)) \tag{6}$$

has its image contained in $(U^*)^{\otimes k-1} \otimes Q$, and coincides with contraction.

There is a map

$$C : J_n^{\langle k \rangle}(M) \to \Lambda^2 U^* \otimes J_n^{\langle k-2 \rangle}(Q \to J_n^1(M)) \tag{7}$$

given by $C(y) = \overline{[\cdot,\cdot]}|_{U_y^{(k-1)}}$. This commutes with the projections, and the zero locus of C consists of the holonomic k-th order jets.

3.3 Let $k > 0$. A *k-th order semi-holonomic differential equation* on n-dimensional submanifolds of M is a subset $R \subseteq J_n^{\langle k \rangle}(M)$. We say that R is smooth if it is a smooth submanifold of $J_n^{\langle k \rangle}(M)$. We say that R is (globally, locally, infinitesimally) differentially closed if it is so considered as a first order equation on submanifolds of $J_n^{\langle k-1 \rangle}(M)$ via the inclusion $J_n^{\langle k \rangle}(M) \subseteq J_n^1(J_n^{\langle k-1 \rangle}(M))$. For $1 \leq j \leq k$, the j-th symbol of a smooth equation R is defined as

$$\tilde{A}_R^j = \ker(\pi_{j,j-1*}|_{\pi_{k,j*}TR}) \subseteq (U^*)^{\otimes j} \otimes Q \tag{8}$$

We also set $\tilde{A}_R^0 = p_Q \pi_{k,0*} TR$, where $p_Q : TM \to Q$ is the projection to the quotient. When $j > k$, we set $\tilde{A}_R^j = (U^*)^{\otimes j-k} \otimes \tilde{A}_R^k$. The total symbol is then defined as $\tilde{A}_R = \bigoplus_{j \geq 0} \tilde{A}_R^j$.

A smooth equation R is said to be j-regular if $\tilde{A}_R^j$ is a smooth vector bundle over R. If R is infinitesimally differentially closed and j-regular for all $1 \leq j \leq k$, then $\tilde{A}_R^k$ is closed under contraction.

When R is infinitesimally differentially closed, we may form the *first semi-holonomic prolongation* of R as $R^{\langle 1 \rangle} = J_n^{\langle k+1 \rangle}(M) \cap J_n^1(R)$. If R is regular in degrees k and $k+1$, this is an infinitesimally differentially closed $(k+1)$-th order semi-holonomic equation, and we have $\tilde{A}_R = \tilde{A}_{R^{\langle 1 \rangle}}$. Moreover, $R^{\langle 1 \rangle} \to R$ is an affine bundle modeled on $\tilde{A}_R^{k+1}$.

More generally, the *l-th semi-holonomic prolongation* of R is defined as $R^{\langle l \rangle} = J_n^{\langle k+l \rangle}(M) \cap J_n^{\langle l \rangle}(R)$, where the intersection is taken inside $J_n^{\langle l \rangle}(J_n^{\langle k \rangle}(M))$. When R is j-regular for all $k \leq j \leq k+l-1$, this coincides with the first prolongation of $R^{\langle l-1 \rangle}$.

When R is infinitesimally differentially closed and k-regular, the map (7) restricted to $R^{\langle 1 \rangle}$ induces a well defined map

$$\mathcal{K}_R : R \to (\Lambda^2 U^* \otimes (TR/\mathcal{C}_n^k|_R))/\delta^1(\tilde{A}_R^{k+1}) \tag{9}$$

called the *total curvature* of R, where

$$\delta^1 : U^* \otimes U^* \otimes ((U^*)^{\otimes k-1} \otimes Q) \to \Lambda^2 U^* \otimes ((U^*)^{\otimes k-1} \otimes Q) \subseteq \Lambda^2 U^* \otimes (TR/\mathcal{C}_n^k|_R) \tag{10}$$

is the map induced by wedge product $U^* \otimes U^* \to \Lambda^2 U^*$. The zero locus of $\mathcal{K}_R$ consists of those holonomic elements of R which extend to a holonomic $(k+1)$-jet.

3.4 Let M be an n-dimensional manifold. Let $J^{\langle k\rangle}_{iso}(M,\mathbb{R}^n)=\pi^{-1}_{k,1}(J^1_{iso}(M,\mathbb{R}^n))$, where $\pi_{k,1}: J^{\langle k\rangle}_n(M\times\mathbb{R}^n)\to J^{\langle 1\rangle}_n(M\times\mathbb{R}^n)$ is the projection. This is the space of *semi-holonomic k-jets of diffeomorphisms* between M and $\mathbb{R}^n$. We may write $J^{\langle k\rangle}_{iso}(M,\mathbb{R}^n)=\tilde{F}^kM\times\mathbb{R}^n$. The space $\tilde{F}^kM$ is called the *k-th order semi-holonomic frame bundle* over M. Observe that the holonomic frame bundle F^kM sits inside $\tilde{F}^kM$. It is easily seen that $\tilde{F}^kM$ is the space of semi-holonomic k-jets of sections of $FM\to M$. In particular, we have that $\tilde{F}^2M=J^1(FM\to M)$.

Consider the space $(\mathbb{R}^{n*})^{\otimes\le k}\otimes\mathbb{R}^n$ of polynomials (in non-commutative variables) of order at most k. Let $\widetilde{GL}^k_n(\mathbb{R})$ be set of polynomials with zero constant term such that the term of order 1 belongs to $GL_n(\mathbb{R})\subseteq\mathbb{R}^*\otimes\mathbb{R}^n$. This forms a Lie group under composition. Observe that the additive group $(\mathbb{R}^{n*})^{\otimes k}\otimes\mathbb{R}^n$ embeds as a closed normal subgroup of $\widetilde{GL}^k_n(\mathbb{R})$ (where the embedding sends P to $P+\mathrm{id}$). We have an exact sequence

$$0\to(\mathbb{R}^{n*})^{\otimes k}\to\widetilde{GL}^k_n(\mathbb{R})\to\widetilde{GL}^{k-1}_n(\mathbb{R})\to 1 \tag{11}$$

There is another, more convenient description of this group. Consider the frame bundle $F\mathbb{R}^n=\mathbb{R}^n\times GL_n(\mathbb{R}^n)$. The group structure in the fibers prolongs to give a group structure in the fibers of $\tilde{F}^k\mathbb{R}^n\to\mathbb{R}^n$. It turns out that the fibers may be identified with $\widetilde{GL}^k_n(\mathbb{R})$. Moreover, we have the following

PROPOSITION 3.4.1. *Let $k\ge 0$. The bundle $\tilde{F}^kM\to M$ is a principal bundle with structure group $\widetilde{GL}^k_n(\mathbb{R})$. Moreover, for $k\ge 2$ we have that $\tilde{F}^kM/((\mathbb{R}^{n*})^{\otimes k}\otimes\mathbb{R}^n)=\tilde{F}^{k-1}M$.*

The holonomic frame bundle F^kM is then a reduction of the structure group of $\tilde{F}^kM$ to the group $GL^k_n(\mathbb{R})$ of k-jets of automorphisms of $\mathbb{R}^n$ at 0.

4 Higher Order G-Structures

4.1 Let $k\ge 0$ and $G\subseteq\widetilde{GL}^k_n(\mathbb{R})$ be a Lie subgroup. A (semi-holonomic) *k-th order G-structure* on an n-dimensional manifold M is a reduction F_G of the structure group of $\tilde{F}^kM$ to G. If F_G is contained inside $\tilde{F}^kM$ then the structure is said to be *holonomic*. It is common in the literature to define higher order structures to be holonomic, however we need the more general concept in order to deal with possibly non-vanishing torsion in the last step of the equivalence method. Indeed, after the first prolongation, $F_{\mathfrak{g}^{(1)}}$ may be thought of as a second order structure on M, which is holonomic if and only if the torsion vanishes. From now on we shall assume all our structures to be semi-holonomic.

Let $\mathfrak{g}$ be the Lie algebra of G. For each $1\le j\le k$, set

$$\tilde{A}^j_{\mathfrak{g}}=\left(\mathfrak{g}\cap(\mathbb{R}^{n*})^{\otimes\ge j}\otimes\mathbb{R}^n\right)/\left(\mathfrak{g}\cap(\mathbb{R}^{n*})^{\otimes\ge j+1}\right)\subseteq(\mathbb{R}^{n*})^{\otimes j}\otimes\mathbb{R}^n \tag{12}$$

In the case $j = 0$, we let $\tilde{A}^j_{\mathfrak{g}} = \mathbb{R}^n$. When $j > k$, let $\tilde{A}^j_{\mathfrak{g}} = (\mathbb{R}^{n*})^{\otimes j-k} \otimes \tilde{A}^k_{\mathfrak{g}}$. Set $\tilde{A}_{\mathfrak{g}} = \bigoplus_{j \geq 0} \tilde{A}^j_{\mathfrak{g}}$.

Observe that $F_G \times \mathbb{R}^n$ is a k-th order equation on sections of $M \times \mathbb{R}^n \to M$, and its total symbol is a trivial bundle with fiber $\tilde{A}_{\mathfrak{g}}$.

For each $0 \leq j \leq k$, let G_j be the image of G under the projection $\tilde{F}^k M \to \tilde{F}^j M$. We have exact sequences

$$0 \to \tilde{A}^j_{\mathfrak{g}} \to G_j \to G_{j-1} \to 0 \tag{13}$$

Let $F_{G_j} = \pi_{k,j} F_G$. This is a j-th order G_j-structure on M. Observe that the vertical distribution of the projection $F_{G_j} \to F_{G_{j-1}}$ is a trivial bundle with fiber $\tilde{A}^j_{\mathfrak{g}}$.

We say that a k-th order structure F_G is *differentially closed* if the corresponding k-th order equation $F_G \times \mathbb{R}^n$ is differentially closed (in this case, the notions of globally, locally and infinitesimally differentially closed coincide). When this happens, $\tilde{A}_{\mathfrak{g}}$ is closed under contraction by vectors in $\mathbb{R}^n$. Observe that first order structures are always differentially closed.

When F_G is differentially closed, we may form its *first (semi-holonomic) prolongation* $F_G^{\langle 1 \rangle}$ which is the bundle over F_G whose fiber over a point y consists of the lifts of $U_y^{(k-1)}$ to $T_y F_G$. Alternatively, we have $F_G^{\langle 1 \rangle} = \tilde{F}^k M \cap J^1_n(F_G)$. This is a $(k+1)$-th order differentially closed structure for the group $G^{\langle 1 \rangle} \subseteq \widetilde{GL}^{k+1}_n(\mathbb{R})$ consisting of those polynomials whose terms of order at most k belong to G, and with the term of order $k+1$ belonging to $\tilde{A}^{k+1}_{\mathfrak{g}}$. Inductively, one may define the *l-th semi-holonomic prolongation* $F_G^{\langle l \rangle}$ as $F_G^{\langle l-1 \rangle \langle 1 \rangle}$.

In general, if F_G is not differentially closed, one may form its clausure as follows. First, replace F_G with $F_{G_1}^{\langle k-1 \rangle} \cap F_G$. Then, replace the new F_G with $F_{G_2}^{\langle k-2 \rangle} \cap F_G$. In general, in the step i we replace F_G with $F_{G_i}^{\langle k-i \rangle} \cap F_G$. At the end of this process, one is left with the largest differentially closed k-th order structure contained in the original one.

4.2 Let F_G be a differentially closed k-th order G-structure on the n-dimensional manifold M. The total curvature $\mathcal{K}_R$ of the semi-holonomic equation $F_G \times \mathbb{R}^n \subseteq J^{\langle k \rangle}_{iso}(M, \mathbb{R}^n)$ restricts to a map

$$\mathcal{K}_{F_G} : F_G \to (\Lambda^2 \mathbb{R}^{n*} \otimes (\mathfrak{g}_{k-1} \oplus \mathbb{R}^n))/\delta^1(\tilde{A}^{k+1}_{\mathfrak{g}}) \tag{14}$$

where $\mathfrak{g}_{k-1}$ is the Lie algebra of G_{k-1} and

$$\delta^1 : \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes ((\mathbb{R}^{n*})^{\otimes k-1} \otimes \mathbb{R}^n) \to \Lambda^2 \mathbb{R}^{n*} \otimes ((\mathbb{R}^{n*})^{\otimes k-1} \otimes \mathbb{R}^n) \tag{15}$$

is the map induced by wedge product. The map $\mathcal{K}_{F_G}$ is called the *total curvature* of F_G.

Example 4.2.1. Let F_G be a first order G-structure and $\nabla : F_G \to F_{G^{\langle 1\rangle}}$ be a section (i.e., a connection). The image of ∇ is a differentially closed second order G-structure (where G embeds into $GL^2_n(\mathbb{R}^n)$ as a subgroup of the homogeneous polynomials of order 1). In this case, $\tilde{A}^{k+1}_{\mathfrak{g}} = 0$, and the total curvature is a map from F_G to $\Lambda^2\mathbb{R}^{n*}\otimes(\mathfrak{g}\oplus\mathbb{R}^n)$. The components of this map are the curvature and the torsion of ∇.

Observe that if $\mathcal{K}_{F_G}$ is constant in the fibers of $\pi_{k,0}|_{F_G}$, we may write $\mathrm{d}\mathcal{K}_{F_G} = \frac{\partial}{\partial\theta^i}\mathcal{K}_{F_G}\theta^i$, where θ^i are the canonical forms on F_G. The derivatives $\frac{\partial}{\partial\theta^i}\mathcal{K}_{F_G}$ are defined on F_{G_1}. In the same way, if the first l derivatives of $\mathcal{K}_{F_G}$ are defined on M, we may form the $(l+1)$-th total derivative $D^{l+1}\mathcal{K}_{F_G}$, which a function on F_{G_1}.

We say that F_G has been *normalized* if $\mathcal{K}_{F_G}$ and all its derivatives are defined on M. If F_G has been normalized, we say that $\mathcal{K}_{F_G}$ *stabilizes at order* K if $D^K\mathcal{K}_{F_G}$ and $D^{K+1}\mathcal{K}_{F_G}$ have (locally) constant rank and both ranks coincide. We may set $K = n$ if we restrict to an open dense subspace of M.

The following proposition restricts the possible values that the lie algebra $\mathfrak{g}$ may take when the structure has been normalized.

Proposition 4.2.2. *Let $G \subseteq \widetilde{GL}^k_n(\mathbb{R})$ be a closed subgroup, and let F_G be a normalized differentially closed G-structure on M. Then $\tilde{A}^j_{\mathfrak{g}} \subseteq S^j\mathbb{R}^{n*}\otimes\mathbb{R}^n$ for all $0 \le j \le n$.*

Proof. Let $1 \le j < k$ and let z, w be two points in $F_{G_{j+1}}$ such that $\pi_{j+1,j}z = \pi_{j+1,j}w$. Then by the semi-holonomic analogue of proposition III.3.2.1, we have

$$0 = C(w) - C(z) = \delta^1(z - w) \tag{16}$$

where $z - w$ is considered as an element of

$$(\mathbb{R}^{n*}\otimes\mathbb{R}^{n*}\otimes(\mathbb{R}^{n*})^{\otimes j-1}) \cap \tilde{A}^{j+1}_{\mathfrak{g}} \tag{17}$$

and C is the map on $J^{\langle j+1\rangle}_{iso}(M,\mathbb{R}^n)$ given by (7). It follows that elements of $\tilde{A}^{j+1}_{\mathfrak{g}}$ are symmetric in the first two entries. By induction, we have the desired result. □

When this happens, we let $\mathcal{A}^j_{\mathfrak{g}} = \tilde{A}^j_{\mathfrak{g}}$ for $j \le k$, and

$$\mathcal{A}^j_{\mathfrak{g}} = (S^{j-k}\mathbb{R}^{n*}\otimes\mathcal{A}^k_{\mathfrak{g}}) \cap (S^j\mathbb{R}^{n*}\otimes\mathbb{R}^n) \tag{18}$$

for $j \ge k$. Then $\mathcal{A}_{\mathfrak{g}} = \bigoplus_{j\ge 0}\mathcal{A}^j_{\mathfrak{g}}$ is a $S\mathbb{R}^{n*}$ comodule, so it makes sense to compute its Spencer cohomology, which shall be denoted by $H(\mathfrak{g})$.

4.3 We now state our main equivalence result for higher order structures. As usual, we shall only conclude the existence of *formal equivalences*, i.e., jets of infinite order of equivalences. The precise meaning of this will be clear from the proof. In the analytic category, this implies the existence of local equivalences.

THEOREM 4.3.1. *Let M, M' be two n-dimensional manifolds. Let F_G, F'_G be two normalized differentially closed k-th order G-structures on M and M' such that the total curvatures $\mathcal{K}_{F_G}$ and $\mathcal{K}_{F'_G}$ stabilize at order K. Let q, q' be two points in M, M' such that $D^{K+1}\mathcal{K}_{F_G}(q) = D^{K+1}\mathcal{K}_{F'_G}(q')$ and the images of $D^{K+1}\mathcal{K}_{F_G}$ and $D^{K+1}\mathcal{K}_{F'_G}$ coincide near $D^{K+1}\mathcal{K}_{F_G}(q)$. If $H^{2,j}(\mathfrak{g}) = 0$ for $j \geq k+2$, there is a formal equivalence between F_G and F'_G sending q to q'.*

Proof. The proof will generalize the proofs of propositions 2.1.1 and 2.2.1. We shall first define a semi-holonomic k-th order differential equation on submanifolds of $M \times M'$, whose solutions are in correspondence with (local) equivalences between the structures. Then we restrict to a submanifold of $M \times M'$ so that the equation becomes holonomic, and use the vanishing of the cohomology to conclude that it is formally integrable.

Let $\psi^1 : FM \times FM' \to J^1_{iso}(M, M')$ be the map which sends each pair of coframes (ω, ω') over a pair $(y, y') \in M \times M'$ to ${\omega'}^{-1}\omega : T_yM \to T_{y'}M'$. From this, one gets an inclusion

$$\tilde{F}^2M \times \tilde{F}^2M' \subseteq J^1_n(FM \times FM' \to M \times M') \tag{19}$$

The restriction of $(\psi^1)^{\langle 1\rangle}$ defines a map

$$\psi^2 : \tilde{F}^2M \times \tilde{F}^2M' \to J^{\langle 2\rangle}_{iso}(M, M') \tag{20}$$

Proceeding inductively, for all j one may construct a G_j-invariant map

$$\psi^j : \tilde{F}^jM \times \tilde{F}^jM' \to J^{\langle j\rangle}_{iso}(M, M') \tag{21}$$

which gives $\tilde{F}^jM \times \tilde{F}^jM'$ the structure of a principal G_j bundle over $J^{\langle j\rangle}_{iso}(M, M')$.

Let R be the image of $\psi^k|_{F_G \times F'_G}$. This is a differentially closed k-th order semi-holonomic equation, regular in all degrees, whose solutions are in correspondence with local equivalences between F_G and F'_G. The pullback of the symbol $\tilde{A}_R$ over ψ^k is a trivial bundle and sits in an exact sequence of $S\mathbb{R}^{n*}$ comodules

$$0 \to \tilde{A}_\mathfrak{g} \to \tilde{A}_\mathfrak{g} \times \tilde{A}_\mathfrak{g} \to \tilde{A}_R \to 0 \tag{22}$$

where the first map is the diagonal. From this we have that $\tilde{A}^j_R \subseteq S^jT^*M \otimes TM'$ for $j \leq k$. Therefore, we may define $\mathcal{A}^j_R = \tilde{A}^j_R$ for $j \leq k$ and

$$\mathcal{A}^j_R = (S^{j-k}T^*M \otimes TM' \otimes \mathcal{A}^k_R) \cap (S^jT^*M \otimes TM') \tag{23}$$

for $j \geq k$. The sum $\mathcal{A}_R = \bigoplus_{j\geq 0} \mathcal{A}^j_R$ is a bundle of ST^*M comodules over R. The pullback of $\mathcal{A}_R$ via ψ^k is a trivial bundle with fiber isomorphic to $\mathcal{A}_\mathfrak{g}$. Therefore, we have that $H^{2,j}(\mathcal{A}_R) = 0$ for all $j \geq k+2$.

Now, let S be the subset of $M \times M'$ where the equality $D^{K+1}\mathcal{K}_{F_G} = D^{K+1}\mathcal{K}_{F'_G}$ holds. We work near (q, q'), so that S is smooth. Let $R|_S = R \cap J^{\langle j\rangle}_n(S)$. The same argument as in proposition 2.2.1 shows that $\pi_{1,0}^{-1}(S) \cap \pi_{k,1}(R) = J_n^1(S) \cap \pi_{k,1}R$. Now, if $z \in \pi_{2,0}^{-1}(S) \cap \pi_{k,2}R$, we have that $U_z^{\langle 1 \rangle}$ is tangent to $J_n^1(S)$ and therefore $z \in J_n^{\langle 2 \rangle}(S) \cap \pi_{k,2}R$. Proceeding inductively, we get $R|_S = \pi_{k,0}^{-1}(S) \cap R$.

From (22) we see that the total curvature of $R|_S$ vanishes. This means that $R|_S \subseteq J_n^k(M \times M')$ is a holonomic equation integrable to first order. Since $H^{2,j}(R|_S) = 0$ for $j \geq k+2$, we conclude that $R|_S$ is formally integrable, and the result follows. □

It is also common to formulate this result requiring that $H^{l,j}(\mathfrak{g})$ vanishes for $l \geq 0$ and $j \geq k+2$. In this case, R is involutive and one may estimate the size of the solution space.

In the case when $\mathcal{A}_{\mathfrak{g}}^j = 0$ for some $j > 0$, the equation R may be prolonged until it becomes a Frobenius system. In this case, one does not need analiticity to guarantee the existence of solutions. This happens, for example, in the case of Riemannian manifolds.

4.4 We now give an full description of Cartan's method for obtaining a complete set of invariants for a first order structure. Let M be an n-dimensional manifold and F_G be a first order G-structure on M.

(i) Suppose that F_G is not normalized, so that for some K, the derivative $D^K\mathcal{K}$ does not descend to M. Choose a normalizing manifold for $D^K\mathcal{K}$ and use this to reduce the structure group to a proper subgroup H. This reduced structure may not be differentially closed. Let j be the least index such that $H_j \subsetneq G_j$. Start again with the j-th order structure F_{H_j}.

(ii) If F_G is normalized, and $H^{2,j}(\mathfrak{g}) = 0$ for $j \geq k+2$ then the derivatives of the curvatures of F_G constitute the desired invariants and the algorithm terminates.

(iii) If F_G is normalized but $H^{2,j}(\mathfrak{g}) \neq 0$ for some $j \geq k+2$, start the algorithm again with the first semi-holonomic prolongation $F_G^{\langle 1 \rangle}$.

Observe that we are assuming, at each step, that normalizing submanifolds exist. This may not be the case, and the method may be generalized to take into account these situations, however we shall not be concerned with this case. Moreover, the algorithm and the invariants obtained depend on the choice of normalizing submanifolds at each step.

In principle, one could need an arbitrary number of prolongations to reach a complete set of invariants, however in practice most problems do not require more than two. In any case, the algorithm is guaranteed to finish in finite time:

THEOREM 4.4.1. *Cartan's algorithm terminates after a finite number of steps.*

Proof. Suppose that we may choose normalizations so that the algorithm does not terminate. Observe that we have to perform step (iii) an infinite number of times. Indeed, each normalization either diminishes the order of the structure or the dimension of the group, so after a finite number of normalization one is forced to go to step (ii) and then to step (iii). Let $F(0), F(1), \ldots$ be the normalized differentially closed structures that we have each time that we go to step (iii) (before prolongation), where $F(i)$ is a $k(i)$-th order structure for the group $G(i)$. Let $\mathfrak{g}(i)$ be the Lie algebra of $G(i)$.

Since $G(i+1)_1 \subseteq G(i)_1$ for all i, there exists m_1 such that $G(i+1)_1 = G(i)_1$ for all $i \geq m_1$. Take m_1 to be the minimum of those integers. Observe that we must have $k(m_1) = 1$ and $k(i) > 1$ for all $i > m_1$. In the same way, there exists $m_2 > m_1$ such that $G(i+1)_2 = G(i)_2$ for all $i \geq m_2$. Taking m_2 to be minimum, we have that $k(m_2) = 2$ and $k(i) > 2$ for $i > m_2$. Inductively, one constructs an increasing sequence m_k such that $G(i+1)_k = G(i)_k$ for all $i \geq m_k$, and furthermore $k(m_k) = k$ and $k(i) > k$ for all $i > m_k$.

Now, $\mathcal{A}_{\mathfrak{g}(m_k)}$ is a decreasing sequence of $S\mathbb{R}^{n*}$ subcomodules of $S\mathbb{R}^{n*} \otimes \mathbb{R}^n$. Since $S\mathbb{R}^n$ is a Noetherian ring, there exists k_0 such that $\mathcal{A}_{\mathfrak{g}(m_{k+1})} = \mathcal{A}_{\mathfrak{g}(m_k)}$ for all $k \geq k_0$. Let $j_0 \geq m_{k_0}$ be a natural number such that $H^{2,j}(\mathcal{A}_{\mathfrak{g}(m_{k_0})}) = 0$ for all $j \geq j_0 + 2$. It follows that the algorithm terminates after reaching $F(j_0)$, which is a contradiction. □

This algorithm may be used to solve the formal equivalence problem for first order G-structures, in the following way. Suppose that we are given two first order G-structures. We run the algorithm on both of them, trying to use the same normalizations for both at each step. If this is not possible, then the two structures cannot be equivalent. If one is able to do this, in the end one arrives a two k-th order structures for which the equivalence problem has the same solutions as the original one. Then (away from singularities) the original problem has a formal solution if and only if the derivatives of the curvatures coincide in the sense given by theorem 4.3.1.

The observation that makes this work is that after each step in the algorithm, the equivalences between F_{G_1} and F'_{G_1} coincide with the equivalences between F_G and F'_G, and moreover they are the same as the equivalences in the initial problem. This does not hold if we start the algorithm with k-th order structures: one may be forced to reduce them to structures of order less than k, which may cause new equivalences to appear. Therefore, this method may not be directly applied for higher order structures. However, if one is given a k-th order G-structure F_G, its first semi-holonomic prolongation $F_G^{\langle 1 \rangle}$ may be interpreted as a first order structure on F_G for the abelian group $\mathcal{A}_{\mathfrak{g}}^{k+1}$. It may be seen that the solutions to an equivalence problem for G-structures are in correspondence with the solutions to the corresponding equivalence problem for $\mathcal{A}_{\mathfrak{g}}^{k+1}$-structures (see 2.2.2 for the case $k = 1$), and so one may always assume that the starting structure is of first order.

Bibliography

[1] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1991.

[2] L. Ehrenpreis, V. W. Guillemin, and S. Sternberg. On Spencer's estimate for δ-Poincaré. *Ann. of Math. (2)*, 82:128–138, 1965.

[3] R. B. Gardner. *The method of equivalence and its applications*, volume 58 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[4] H. Goldschmidt. Existence theorems for analytic linear partial differential equations. *Ann. of Math. (2)*, 86:246–270, 1967.

[5] H. Goldschmidt. Integrability criteria for systems of nonlinear partial differential equations. *J. Differential Geometry*, 1:269–307, 1967.

[6] V. W. Guillemin and S. Sternberg. An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.*, 70:16–47, 1964.

[7] M. Kuranishi. On E. Cartan's prolongation theorem of exterior differential systems. *Amer. J. Math.*, 79:1–47, 1957.

[8] H. Lewy. An example of a smooth linear partial differential equation without solution. *Ann. of Math. (2)*, 66:155–158, 1957.

[9] B. Malgrange. Cartan involutiveness = Mumford regularity. In *Commutative algebra (Grenoble/Lyon, 2001)*, volume 331 of *Contemp. Math.*, pages 193–205. Amer. Math. Soc., Providence, RI, 2003.

[10] P. J. Olver. *Equivalence, invariants, and symmetry.* Cambridge University Press, Cambridge, 1995.

[11] D. C. Spencer. Deformation of structures on manifolds defined by transitive, continuous pseudogroups. I. Infinitesimal deformations of structure. *Ann. of Math. (2)*, 76:306–398, 1962.

[12] D. C. Spencer. Deformation of structures on manifolds defined by transitive, continuous pseudogroups. II. Deformations of structure. *Ann. of Math. (2)*, 76:399–445, 1962.

[13] W. J. Sweeney. "The δ-Poincaré estimate". *Pacific J. Math.*, 20:559–570, 1967.