CATEGORIFICATION OF SHEAF THEORY

G. STEFANICH

Abstract. We discuss a systematic procedure for categorifying presentable six functor formalisms. Our main result produces, given the input of a representation of the ∞-category of correspondences of an ∞-category with finite limits $\mathcal{C}$, a compatible sequence of representations of the (∞, n)-category of correspondences of $\mathcal{C}$ for every $n \geq 1$. As an application, we explain a general recipe for constructing topological field theories.

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1. Introduction

Let X be an algebraic stack over a base ring k, and let $\mathrm{QCoh}(X)$ be the stable ∞-category of quasi-coherent sheaves on X. In good cases, for each closed manifold M we have an identification

$$\int_M \mathrm{QCoh}(X) = \mathrm{QCoh}(\mathrm{Maps}(M, X))$$

where the left hand side denotes the factorization homology of $\mathrm{QCoh}(X)$ regarded as an E_∞-algebra in presentable stable k-linear ∞-categories, and $\mathrm{Maps}(M, X)$ denotes the derived moduli stack parametrizing locally constant maps from M into X.

The fundamental ingredient needed in order to make the above factorization homology computation is the categorical Künneth formula: in good cases, for every pair of maps of algebraic stacks $Y \to S$ and $Z \to S$ one has an identification

$$\mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(Z) = \mathrm{QCoh}(Y \times_S Z).$$

This is the case for instance when all the stacks involved are perfect in the sense of [BZFN10]; this includes for instance schemes, and many algebraic stacks in characteristic zero.

When working with more general sheaf theories, the categorical Künneth formula often tends to fail. This is the case for instance in the theories of ind-coherent sheaves and étale sheaves. Even for quasi-coherent sheaves, when working in positive characteristic the categorical Künneth formula is known only under fairly restrictive conditions, and we expect

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that it in fact often fails, given the known pathologies for QCoh in that setting [Ste25]. In light of this, it is not surprising that the more general formula

$$\int_M \mathrm{Sh}(X) = \mathrm{Sh}(\mathrm{Maps}(M, X))$$

often fails. Examples are not hard to find: if Sh = IndCoh then the formula is false already when M is a circle and X is the affine line.

The goal of this note is to present a framework that, among other things, allows one to obtain a replacement of the factorization homology formula for a general sheaf theory Sh. The general statement will not (and cannot) be about factorization homology; instead, we will obtain $\mathrm{Sh}(\mathrm{Maps}(M, X))$ as the value on M of a topological field theory.

Sheaf-theoretic topological field theories. The connection between factorization homology and topological field theory is given by [Sch14]: factorization homology provides the values of topological field theories valued in Morita higher categories. Thus one can recast the original QCoh-factorization homology formula as follows: for good enough X, there is an equivalence

$$\chi_{n\mathrm{Mor},\mathrm{QCoh},X}(M) = \mathrm{QCoh}(\mathrm{Maps}(M, X))$$

where

$$\chi_{n\mathrm{Mor},\mathrm{QCoh},X} : n\mathrm{Cob} \to n\mathrm{Mor}(\mathrm{Pr}_{\mathrm{st},k})$$

is a fully extended unoriented topological field theory valued in the Morita theory of E_n-monoidal presentable stable k-linear ∞-categories, with n being the dimension of M. We remark that although the source of $\chi_{n\mathrm{Mor},\mathrm{QCoh},X}$ involves cobordisms of dimension at most n, we may secretly think of this as being an $(n+2)$-dimensional topological field theory, given the categorical complexity of its outputs: indeed, its values on manifolds of dimension n are not numbers but ∞-categories.

In the same way that the factorization homology formula does not work for general sheaf theories, we should not expect to obtain $\mathrm{Sh}(\mathrm{Maps}(M, X))$ as the value of a topological field theory valued in E_n-monoidal presentable ∞-categories. Instead, our topological field theories will be valued in presentable higher categories [Ste20b].

Up to size issues, one may think about a presentable (∞, n)-category inductively as being an (∞, n)-category which has colimits and whose Homs are presentable $(\infty, n-1)$-categories. The totality of all presentable (∞, n)-categories forms a symmetric monoidal $(\infty, n+1)$-category $n\mathrm{Pr}$. Imposing stability and k-linearity at the level of $(n-1)$-cells yields a variant $n\mathrm{Pr}_{\mathrm{st},k}$ which receives a symmetric monoidal functor

$$n\mathrm{Mod} : n\mathrm{Mor}(\mathrm{Pr}_{\mathrm{st},k}) \to (n+1)\mathrm{Pr}_{\mathrm{st},k}\,.$$

Already for QCoh, the passage from $n\mathrm{Mor}(\mathrm{Pr}_{\mathrm{st},k})$ to $(n+1)\mathrm{Pr}_{\mathrm{st},k}$ has the benefit of allowing one to formulate a result which places no restrictions on the stacks:

Theorem 1.1. *For every algebraic stack X over k and every $n \geq 0$, there is a symmetric monoidal functor $\chi_{(n+1)\mathrm{QCoh},X} : n\mathrm{Cob} \to (n+1)\mathrm{Pr}_{\mathrm{st},k}$ such that for every closed manifold M of dimension n we have an equivalence*

$$\chi_{(n+1)\mathrm{QCoh},X}(M) = \mathrm{QCoh}(\mathrm{Maps}(M, X)).$$

The value of $\chi_{(n+1)\mathrm{QCoh},X}$ on the point is given by the $(\infty, n+1)$-category $(n+1)\mathrm{QCoh}(X)$ of quasi-coherent sheaves of (∞, n)-categories on X which we introduced in [Ste21], and the above theorem is a direct consequence of the functoriality properties of $(n+1)\mathrm{QCoh}$ together with [CHS25] theorem D. From this perspective, the role of the categorical Künneth formulas in factorization homology computations is played by the base change formulas in the theory of sheaves of higher categories.

The connection between $\chi_{(n+1)\mathrm{QCoh},X}$ and $\chi_{n\mathrm{Mor},\mathrm{QCoh},X}$ is given by the 1-affineness theorems of [Gai15, Ste21]: for X good enough, one has an equivalence

$$(n+1)\mathrm{QCoh}(X) = n\mathrm{Mod}_{\mathrm{QCoh}(X)}\,.$$

As shown in [Ste25], 1-affineness fails often beyond characteristic zero, so although theorem 1.1 still holds, it cannot be rephrased in Morita theory terms.

The output of this note is a theory that allows one to generalize theorem 1.1 beyond the quasi-coherent setting. In fact, there is nothing special about the setting of algebraic stacks: one may work on a general ∞-category with finite limits:

Theorem 1.2. *Let $\mathcal{C}$ be an ∞-category with finite limits and let* Sh *be a presentable six functor formalism on $\mathcal{C}$ (see 1 below). Then there exists a sequence of presentable symmetric monoidal $(\infty, n+1)$-categories $\mathcal{T}_n$ with $\mathrm{End}_{\mathcal{T}_n}(1_{\mathcal{T}_n}) = \mathcal{T}_{n-1}$ for all $n \geq 1$, with the following feature: for every object X in $\mathcal{C}$ and every $n \geq 0$ there is a symmetric monoidal functor $\chi_{(n+1)\mathrm{Sh},X} : n\mathrm{Cob} \to \mathcal{T}_{n+1}$ such that for each closed manifold M of dimension n there is an equivalence*

$$\Gamma(\chi_{(n+1)\mathrm{Sh},X}(M)) = \mathrm{Sh}(\mathrm{Maps}(M, X))$$

where $\Gamma(-) = \mathrm{Hom}_{\mathcal{T}_1}(1_{\mathcal{T}_1,-})$ denotes the passage to the underlying ∞-category.

Theorem 1.2 works formally just like theorem 1.1 and is a direct consequence of the existence of a family of categorifications $n\mathrm{Sh}$ of Sh with good functoriality properties, which is what this note aims to establish. The only extra twist beyond the quasi-coherent setting is that the targets $\mathcal{T}_n$ may be somewhat exotic. One in fact has that $\mathcal{T}_n = (n+1)\mathrm{Sh}(1_{\mathcal{C}})$, and for each closed manifold M of dimension $0 \leq d \leq n$, the value of $\chi_{(n+1)\mathrm{Sh},X}$ on M is given by a canonical upgrade of $(n+1-d)\mathrm{Sh}(X)$ to $(n+2-d)\mathrm{Sh}(1_{\mathcal{C}})$.

As a specialization of theorem 1.2 one obtains the following:

Corollary 1.3. *Let X be a smooth scheme over a field k of characteristic zero. Then there exists a symmetric monoidal functor $\chi_{2\mathrm{IndCoh},X} : 1\mathrm{Cob} \to 3\mathrm{IndCoh}(\mathrm{Spec}(k))$ such that*

$$\chi_{2\mathrm{IndCoh},X}(S^1) = \mathrm{IndCoh}(\mathrm{Maps}(S^1, X)) = \mathrm{QCoh}(T^*[2]X).$$

The topological field theory from corollary 1.3 may be thought of as a mathematical incarnation of the Rozansky-Witten theory of T^*X [KRS09]. More precisely, $\chi_{2\mathrm{IndCoh},X}$ satisfies the design criteria for a conical version of Rozansky-Witten theory; non-conical versions can also be obtained after a suitable two-periodization of this construction.

The ∞-category $\mathrm{IndCoh}(\mathrm{Maps}(S^1, X))$ arises not only as the value on S^1 of the theory, but also as the center: $\mathrm{IndCoh}(\mathrm{Maps}(S^1, X))$ has a canonical acion on $2\mathrm{IndCoh}(X)$. Via Koszul duality, this allows one to express $2\mathrm{IndCoh}(X)$ as the global sections of a sheaf of $(\infty, 2)$-categories on T^*X with its conical Zariski topology. In other words, the theory of ind-coherent sheaves of categories on X is not only local on X, but also admits a microlocal theory, which may be thought of as a categorification of the microlocal theory of ind-coherent sheaves developed in [AG15].

We also have the following specialization of 1.2 which is relevant in four dimensions:

Corollary 1.4. *Let G be an affine algebraic group over a field k of characteristic zero. Then there exists a symmetric monoidal functor $\chi_{3\mathrm{IndCoh},\mathrm{B}G} : 2\mathrm{Cob} \to 4\mathrm{IndCoh}(\mathrm{Spec}(k))$ such that for every closed manifold M of dimension 2 we have an equivalence*

$$\chi_{3\mathrm{IndCoh},\mathrm{B}G}(M) = \mathrm{IndCoh}(\mathrm{Maps}(M, X)).$$

Corollary 1.4 is consistent with the expectation that the various structures arising in (the Betti form of) the geometric Langlands program may be organized into the framework of 4-dimensional topological field theory [KW07, BZN18]. More precisely, it provides a realization of a version of the spectral Betti geometric Langlands TQFT with no restriction on the central parameters (that is, no conditions of nilpotent singular support). A version of corollary 1.4 with nilpotent singular support is the subject of upcoming work.

Categorification of sheaf theories. Let $\mathcal{C}$ be an ∞-category with finite limits. Associated to $\mathcal{C}$ there is a symmetric monoidal ∞-category $\mathrm{Corr}(\mathcal{C})$ called the ∞-category of correspondences, with the following features:

- The anima of objects of $\mathrm{Corr}(\mathcal{C})$ agrees with the anima of objects of $\mathcal{C}$.
- Let X and Y be a pair of objects of $\mathcal{C}$. Then a morphism from X to Y in $\mathrm{Corr}(\mathcal{C})$ consists of a span $X \leftarrow S \to Y$ in $\mathcal{C}$.
- Let $X \leftarrow S \to Y$ and $Y \leftarrow T \to Z$ be a pair of spans in $\mathcal{C}$, which we regard as morphisms in $\mathrm{Corr}(\mathcal{C})$. Then their composition is given by the span $X \leftarrow S \times_Y T \to Z$.
- Let X and Y be a pair of objects of $\mathcal{C}$. Then their tensor product in $\mathrm{Corr}(\mathcal{C})$ is given by $X \times Y$.

As explained in [GR17], the ∞-category $\mathrm{Corr}(\mathcal{C})$ (and variants of it) provides a convenient way of capturing the functoriality present in various sheaf theories of interest. More precisely, if $\mathcal{T}$ is a symmetric monoidal $(\infty, 2)$-category whose objects we think about as being ∞-categories of some sort, then a lax symmetric monoidal functor $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathcal{T}$ gives rise to the following data:

- For each object X in $\mathcal{C}$ an object $\mathrm{Sh}(X)$ in $\mathcal{T}$ which we think of as the ∞-category of sheaves on X.
- For each map $f : X \to Y$ in $\mathcal{C}$ a pair of morphisms $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ and $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ which we think of as pullback and pushforward functors.
- For each cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

  in $\mathcal{C}$, an isomorphism $f'_! g'^* = g^* f_!$.
- For each pair of objects X, Y in $\mathcal{C}$, a morphism $\boxtimes : \mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y)$ which we think about as the exterior tensor product functor.

Lax symmetric monoidal functors out of $\mathrm{Corr}(\mathcal{C})$ are called three functor formalisms, and requiring right adjoints to $f_!$, f^* and $\mathcal{F} \boxtimes -$ one arrives at the notion of a six functor formalism [Man22, Sch23]. The existence of these extra adjoints is automatic when the target is the $(\infty, 2)$-category Pr of presentable ∞-categories; in this case a lax symmetric monoidal functor $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ is called a presentable six functor formalism.

As explained in [Ste20a, Ste21], when working with theories of sheaves of higher categories one tends to encounter much stronger functoriality properties than in ordinary sheaf theory.

In this context, the role of $\mathrm{Corr}(\mathcal{C})$ is played by the symmetric monoidal (∞, n)-category $n\mathrm{Corr}(\mathcal{C})$, which in the case $n = 1$ agrees with $\mathrm{Corr}(\mathcal{C})$ and for $n > 1$ admits the following informal inductive description:

- The anima of objects of $n\mathrm{Corr}(\mathcal{C})$ agrees with the anima of objects of $\mathcal{C}$.
- Let X and Y be a pair of objects of $\mathcal{C}$. Then the $(\infty, n-1)$-category of morphisms from X to Y in $n\mathrm{Corr}(\mathcal{C})$ is given by $(n-1)\mathrm{Corr}({}_{X\backslash}\mathcal{C}_{/Y})$.
- Let X and Y be a pair of objects of $\mathcal{C}$. Then their tensor product in $n\mathrm{Corr}(\mathcal{C})$ is given by $X \times Y$.

Our main result constructs, for every presentable six functor formalism $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$, a compatible sequence of categorified formalisms $n\mathrm{Sh} : n\mathrm{Corr}(\mathcal{C}) \to n\mathrm{Pr}$ which we think about as providing theories of sheaves of higher categories of flavor Sh. More precisely, we have the following:

Theorem 1.5. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Then there exists:*

- *A sequence of presentable symmetric monoidal $(\infty, n+1)$-categories $\mathcal{T}_n$ with $\mathrm{End}_{\mathcal{T}_n}(1_{\mathcal{T}_n}) = \mathcal{T}_{n-1}$ for all $n \geq 1$.*
- *A sequence of symmetric monoidal functors $n\mathrm{Sh}^\sharp : n\mathrm{Corr}(\mathcal{C}) \to \mathcal{T}_n$ such that the resulting square*

$$\begin{array}{ccc} \mathrm{End}_{n\mathrm{Corr}(\mathcal{C})}(1_{\mathcal{C}}) & \xrightarrow{n\mathrm{Sh}^\sharp} & \mathrm{End}_{\mathcal{T}_n}(1_{\mathcal{T}_n}) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ (n-1)\mathrm{Corr}(\mathcal{C}) & \xrightarrow{(n-1)\mathrm{Sh}^\sharp} & \mathcal{T}_{n-1} \end{array}$$

commutes, and with the feature that $\mathrm{Sh}(-) = \mathrm{Hom}_{\mathcal{T}_1}(1_{\mathcal{T}_1}, 1\mathrm{Sh}^\sharp(-))$.

Furthermore, the cells in $\mathcal{T}_n$ are generated under (weighted) colimits by cells in the image of $n\mathrm{Sh}^\sharp$, and the unique symmetric monoidal functor $n\mathrm{Pr} \to \mathcal{T}_n$ admits a colimit preserving right adjoint $\Gamma(-)$.

In the language of [Ste21], the sequence of higher presentable categories $\mathcal{T}_n$ assembles into a categorical spectrum, and the sequence of functors $n\mathrm{Sh}^\sharp$ assembles into a representation of the categorical spectrum of correspondences of $\mathcal{C}$.

Although in general the targets $\mathcal{T}_n$ are different from $n\mathrm{Pr}$, one may obtain (non symmetric monoidal) functors valued in $n\mathrm{Pr}$ by setting $n\mathrm{Sh}(-) = \Gamma(n\mathrm{Sh}(-)^\sharp)$. In the case $n = 1$ this recovers the starting sheaf theory. In general, the conditions from the statement of theorem 1.5 imply the following inductive description:

- $(n+1)\mathrm{Sh}(X)$ is a presentable $(\infty, n+1)$-category freely generated under weighted colimits by the objects $n\mathrm{Sh}(Y/X)$ which are obtained by pushforward along a map $Y \to X$ of the unit of $(n+1)\mathrm{Sh}(Y)$.
- Given a pair of maps $Y \to X$ and $Z \to X$ one has

$$\mathrm{Hom}(n\mathrm{Sh}(Y/X), n\mathrm{Sh}(Z/X)) = n\mathrm{Sh}(Y \times_X Z).$$

In general, the (∞, n)-category $n\mathrm{Sh}(X)$ is rather complex: the above description shows that it has generators which are parametrized by maps $Y \to X$. It is however often possible to show that a smaller collection of generators suffice. We refer to section 4 for a discussion of the basic computational toolset that allows this.

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1.7. **Conventions and notation.** In the remainder of this text we follow the convention where the word category stands for ∞-category, and use the term n-category to refer to (∞, n)-categories. We also use the terms scheme and stack to refer to spectral schemes and stacks.

We make use throughout this text of the theory of enriched (higher) categories. For each possibly large monoidal category $\mathcal{M}$ we denote by $\mathrm{Cat}^{\mathcal{M}}$ the category of $\mathcal{M}$-enriched categories with a small anima of objects. In the case when $\mathcal{M}$ is symmetric monoidal, $\mathrm{Cat}^{\mathcal{M}}$ has an induced symmetric monoidal structure, so it makes sense to define $n\mathrm{Cat}^{\mathcal{M}}$ inductively for all $n \geq 1$ as follows:

- If $n = 1$ then $n\mathrm{Cat}^{\mathcal{M}} = \mathrm{Cat}^{\mathcal{M}}$.
- If $n > 1$ then $n\mathrm{Cat}^{\mathcal{M}} = \mathrm{Cat}^{(n-1)\mathrm{Cat}^{\mathcal{M}}} = (n-1)\mathrm{Cat}^{\mathrm{Cat}^{\mathcal{M}}}$.

Objects of $n\mathrm{Cat}^{\mathcal{M}}$ are called $\mathcal{M}$-enriched n-categories. In the special case when $\mathcal{M}$ is the category of small anima we set $n\mathrm{Cat} = n\mathrm{Cat}^{\mathcal{M}}$, and call it the category of n-categories.

The assignment $\mathcal{M} \mapsto \mathrm{Cat}^{\mathcal{M}}$ is functorial in lax monoidal functors in $\mathcal{M}$. Given such a functor $F : \mathcal{M} \to \mathcal{M}'$ we denote by $F_! : \mathrm{Cat}^{\mathcal{M}} \to \mathrm{Cat}^{\mathcal{M}'}$ the induced map. In the case when F is a (lax) symmetric monoidal functor between symmetric monoidal categories we have that $F_!$ is also (lax) symmetric monoidal.

If A is an algebra in a monoidal category $\mathcal{M}$ we denote by BA the corresponding $\mathcal{M}$-enriched category. The assignment $A \mapsto BA$ provides a fully faithful embedding from the category of algebras in $\mathcal{M}$ into the category of pointed $\mathcal{M}$-enriched categories. We denote by Ω its right adjoint; in other words, this is the functor which sends a pointed $\mathcal{M}$-enriched category $(\mathcal{D}, X_0)$ to the algebra of endomorphisms of X_0.

In the case when $\mathcal{M}$ is symmetric monoidal, the assignment $A \mapsto BA$ has a symmetric monoidal structure. In particular, if A is a commutative algebra then BA has the structure of symmetric monoidal $\mathcal{M}$-enriched category. Iterating the construction $A \mapsto BA$ we may thus define for each $n \geq 1$ a symmetric monoidal $\mathcal{M}$-enriched n-category $B^n A$. The assignment $A \mapsto B^n A$ has a right adjoint which we denote by Ω^n.

2. Enrichment of $n\mathrm{Corr}(\mathcal{C})$

Let $\mathcal{C}$ be a category with finite limits. Then every object of $\mathrm{Corr}(\mathcal{C})$ is self dual. In particular, the symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C})$ is closed, so that $\mathrm{Corr}(\mathcal{C})$ has a canonical structure of $\mathrm{Corr}(\mathcal{C})$-enriched category. The goal of this section is to show that, more generally, $n\mathrm{Corr}(\mathcal{C})$ may be given the structure of a $\mathrm{Corr}(\mathcal{C})$-enriched n-category for all $n \geq 1$.

Notation 2.1. Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let A be a commutative algebra in $\mathcal{M}$. Let $G : \mathrm{Mod}_A(\mathcal{M}) \to \mathcal{M}$ be the forgetful functor, which we equip with its canonical lax symmetric monoidal structure. Consider the induced lax symmetric monoidal functor

$$G_! : \mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})} \to \mathrm{Cat}^{\mathcal{M}} .$$

This induces a lax symmetric monoidal functor

$$\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})} = \mathrm{Mod}_{1_{\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})}}}(\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})}) \to \mathrm{Mod}_{G_! 1_{\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})}}}(\mathrm{Cat}^{\mathcal{M}}).$$

Observe that $G_! 1_{\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})}} = BA$ as symmetric monoidal $\mathcal{M}$-enriched categories. We denote by

$$\Gamma_A : \mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})} \to \mathrm{Mod}_{BA}(\mathrm{Cat}^{\mathcal{M}})$$

the induced lax symmetric monoidal functor.

Proposition 2.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let A be a commutative algebra in $\mathcal{M}$. Then the lax symmetric monoidal functor Γ_A from notation 2.1 is a symmetric monoidal equivalence.*

Proof. Let $\mathrm{Algbrd}(\mathcal{M})$ (resp. $\mathrm{Algbrd}(\mathrm{Mod}_A(\mathcal{M}))$) be the category of $\mathcal{M}$-algebroids (resp. $\mathrm{Mod}_A(\mathcal{M})$-algebroids); that is, these are the categories of non-necessarily univalent enriched categories. Let $B^{\mathrm{pre}}A$ be the $\mathcal{M}$-algebroid with a single object and endomorphisms A, and note that we have a lax symmetric monoidal functor

$$\Gamma_A^{\mathrm{pre}} : \mathrm{Algbrd}(\mathrm{Mod}_A(\mathcal{M})) \to \mathrm{Mod}_{B^{\mathrm{pre}}(A)}(\mathrm{Algbrd}(\mathcal{M}))$$

defined similarly to Γ_A. The proposition will follow if we show that Γ_A^{pre} is a symmetric monoidal equivalence. Let $G_!^{\mathrm{pre}} : \mathrm{Algbrd}(\mathrm{Mod}_A(\mathcal{M})) \to \mathrm{Algbrd}(\mathcal{M})$ be the lax symmetric monoidal functor induced by G. Note that $G_!^{\mathrm{pre}}$ is right adjoint to the symmetric monoidal functor $F_!^{\mathrm{pre}}$ induced from F. A monadicity argument reduces us to proving the following:

- (i) $G_!^{\mathrm{pre}}$ commutes strictly with the action of $\mathrm{Algbrd}(\mathcal{M})$.
- (ii) Let $S_\bullet$ be a simplicial diagram in $\mathrm{Algbrd}(\mathrm{Mod}_A(\mathcal{M}))$ and assume that $G_!^{\mathrm{pre}} S_\bullet$ is the simplicial Bar resolution of some $B^{\mathrm{pre}}A$-module in $\mathrm{Algbrd}(\mathcal{M})$. Then $G_!^{\mathrm{pre}}$ preserves the geometric realization of $S_\bullet$.
- (iii) $G_!^{\mathrm{pre}}$ is conservative.

Item (iii) follows readily from the fact that G is conservative, while item (i) follows from the fact that G commutes strictly with the action of $\mathcal{M}$. It remains to establish (ii). Since the anima of objects of $B^{\mathrm{pre}}A$ is a singleton the simplicial anima underlying $G_!^{\mathrm{pre}} S_\bullet$ is constant. It follows that the simplicial anima underlying $S_\bullet$ is constant. Consequently, we may reduce to showing that for each anima J the functor $\mathrm{Algbrd}_J(\mathrm{Mod}_A(\mathcal{M})) \to \mathrm{Algbrd}_J(\mathcal{M})$ induced by G on algebroids with anima of objects J preserves geometric realizations. This follows from the fact that G itself preserves geometric realizations. □

Notation 2.3. Let $n \geq 1$. Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let A be a commutative algebra in $\mathcal{M}$. We define a lax symmetric monoidal functor

$$\Gamma_A^n : n\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})} \to \mathrm{Mod}_{B^n A}(n\mathrm{Cat}^{\mathcal{M}})$$

by induction on n as follows:

- If $n = 1$ we let Γ_A^n be the lax symmetric monoidal functor Γ_A from notation 2.1.
- Assume that $n > 1$. Then we let Γ_A^n be given by the composition

$$\begin{aligned} n\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})} = \mathrm{Cat}^{(n-1)\mathrm{Cat}^{\mathrm{Mod}_A(\mathcal{M})}} &\xrightarrow{(\Gamma_A^{n-1})_!} \mathrm{Cat}^{\mathrm{Mod}_{B^{n-1}A}((n-1)\mathrm{Cat}^{\mathcal{M}})} \\ &\xrightarrow{\Gamma_{B^{n-1}A}} \mathrm{Mod}_{B^n A}(\mathrm{Cat}^{(n-1)\mathrm{Cat}^{\mathcal{M}}}) \\ &= \mathrm{Mod}_{B^n A}(n\mathrm{Cat}^{\mathcal{M}}) \end{aligned}$$

Corollary 2.4. *Let $n \geq 1$. Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let A be a commutative algebra in $\mathcal{M}$. Then the lax symmetric monoidal functor Γ_A^n from notation 2.3 is a symmetric monoidal equivalence.*

Proof. Follows from an inductive application of proposition 2.2. □

Definition 2.5. Let $\mathcal{A}$ be a symmetric monoidal category. We say that an $\mathcal{A}$-module category $\mathcal{D}$ is closed if it admits all Hom objects. We denote by $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})_{\mathrm{closed}}$ the full subcategory of $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})$ on the closed $\mathcal{A}$-modules.

Remark 2.6. Let $\mathcal{A}$ be a symmetric monoidal category, and assume that every object of $\mathcal{A}$ is dualizable. Then $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})_{\mathrm{closed}}$ is closed under tensor products inside $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})$. Furthermore, the procedure of $\mathcal{A}$-enrichment of closed $\mathcal{A}$-modules assembles into a fully faithful symmetric monoidal functor $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})_{\mathrm{closed}} \to \mathrm{Cat}^{\mathcal{A}}$.

Definition 2.7. Let $n \geq 1$ and let $\mathcal{A}$ be a symmetric monoidal category, which we regard as a commutative algebra in $\mathcal{M} = \mathrm{Cat}$. Assume that every object of $\mathcal{A}$ is dualizable. We say that a $B^{n-1}\mathcal{A}$-module $\mathcal{D}$ in $n\mathrm{Cat}$ is closed if the inverse image of $\mathcal{D}$ under $\Gamma_{\mathcal{A}}^{n-1}$ belongs to $(n-1)\mathrm{Cat}^{\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})_{\mathrm{closed}}} \subseteq (n-1)\mathrm{Cat}^{\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cat})}$.

Definition 2.7 provides a way of equipping an n-category $\mathcal{D}$ with an enrichment over a symmetric monoidal category $\mathcal{A}$ such that all objects of $\mathcal{A}$ dualizable: it is enough to equip $\mathcal{D}$ with the structure of a closed module over $B^{n-1}A$. We now apply this to our case of interest:

Proposition 2.8. *Let $\mathcal{C}$ be a category with finite limits and let $n \geq 2$. Consider the symmetric monoidal functor*

$$B^{n-1}\,\mathrm{Corr}(\mathcal{C}) = B^{n-1}\Omega^{n-1} n\mathrm{Corr}(\mathcal{C}) \to n\mathrm{Corr}(\mathcal{C})$$

obtained from the counit of the $B^n \dashv \Omega^n$ adjunction. Then the induced $B^{n-1}\,\mathrm{Corr}(\mathcal{C})$-module structure on $n\mathrm{Corr}(\mathcal{C})$ is closed.

The closure conditions needed to establish proposition 2.8 all follow from the following:

Proposition 2.9. *Let $\mathcal{C}$ be a category with finite limits. Let X be an object of $\mathcal{C}$ and consider the canonical action of $\mathrm{Corr}(\mathcal{C})$ on $\mathrm{Corr}(\mathcal{C}_{/X})$. Then $\mathrm{Corr}(\mathcal{C}_{/X})$ is a closed module over $\mathrm{Corr}(\mathcal{C})$.*

Proof. The module structure arises by restriction of scalars from the symmetric monoidal functor $\alpha : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Corr}(\mathcal{C}_{/X})$ induced by the functor $\mathcal{C} \to \mathcal{C}_{/X}$ of product with X. Since α admits a right adjoint, we may reduce to showing that $\mathrm{Corr}(\mathcal{C}_{/X})$ is closed as a module over itself. Indeed, this follows from the fact that every object in a category of correspondences is dualizable. □

Notation 2.10. Let $\mathcal{C}$ be a category with finite limits and let $n \geq 2$. It follows from proposition 2.8 that the inverse image of $n\mathrm{Corr}(\mathcal{C})$ under $\Gamma_{\mathrm{Corr}(\mathcal{C})}^{n-1}$ belongs to $(n-1)\mathrm{Cat}^{\mathrm{Mod}_{\mathrm{Corr}(\mathcal{C})}(\mathrm{Cat})_{\mathrm{closed}}}$. We let $n\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C})$ be its image under the functor

$$(n-1)\mathrm{Cat}^{\mathrm{Mod}_{\mathrm{Corr}(\mathcal{C})}(\mathrm{Cat})_{\mathrm{closed}}} \to (n-1)\mathrm{Cat}^{\mathrm{Cat}^{\mathrm{Corr}(\mathcal{C})}} = n\mathrm{Cat}^{\mathrm{Corr}(\mathcal{C})}$$

induced from the symmetric monoidal inclusion $\mathrm{Mod}_{\mathrm{Corr}(\mathcal{C})}(\mathrm{Cat})_{\mathrm{closed}} \to \mathrm{Cat}^{\mathrm{Corr}(\mathcal{C})}$ (see remark 2.6). We extend this notation also to the case $n = 1$ by letting $\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C})$ be the $\mathrm{Corr}(\mathcal{C})$-enriched category associated to the closed symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C})$.

3. Construction of $n\mathrm{Sh}(X)$

We now turn to the proof of theorem 1.5. We will construct the maps

$$n\mathrm{Sh}^\sharp : n\mathrm{Corr}(\mathcal{C}) \to \mathcal{T}_n$$

as functors of pointed n-categories. The compatibilities between different values of n will be manifest from the construction. Note that the symmetric monoidal structures on $\mathcal{T}_n$ and $n\mathrm{Sh}$ follow from these compatibilities.

Notation 3.1. Let $\mathcal{M}$ be a symmetric monoidal category. We denote by $\Gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{An}$ the functor corepresented by the unit of $\mathcal{M}$. We equip $\Gamma_{\mathcal{M}}$ with the induced lax symmetric monoidal structure.

Remark 3.2. Let $\mathcal{M}$ be a symmetric monoidal category. Then $\Gamma_{\mathcal{M}}$ is the initial lax symmetric monoidal functor from $\mathcal{M}$ into anima.

Notation 3.3. Let $n \geq 1$. We let

$$n\mathrm{Mod} : n\mathrm{Cat}^{\mathrm{Pr}} \to (n+1)\mathrm{Pr}$$

be the canonical functor. In other words, $n\mathrm{Mod}$ is obtained by freely adding colimits of cells of dimension $0 \leq d \leq n$. We note that for each object $\mathcal{D}$ in $n\mathrm{Cat}^{\mathrm{Pr}}$ we have a morphism $Y_{\mathcal{D}} : \mathcal{D} \to n\mathrm{Mod}_{\mathcal{D}}$.

Construction 3.4. Let $\mathcal{C}$ be a category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Let $\eta : \Gamma_{\mathrm{Corr}(\mathcal{C})} \to \mathrm{Sh}$ be the canonical lax symmetric monoidal natural transformation. For each $n > 1$, we have that η induces a functor

$$n\mathrm{Sh}^{\sharp,\mathrm{pre}} : n\mathrm{Corr}(\mathcal{C}) = (\Gamma_{\mathrm{Corr}(\mathcal{C})})_! n\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C}) \to \mathrm{Sh}_! \, n\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C}).$$

We let $\mathcal{T}_n = n\mathrm{Mod}_{\mathrm{Sh}_! \, n\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C})}$, and let $n\mathrm{Sh}^\sharp$ be the composite functor

$$n\mathrm{Corr}(\mathcal{C}) \xrightarrow{n\mathrm{Sh}^{\sharp,\mathrm{pre}}} \mathrm{Sh}_! \, n\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C}) \xrightarrow{Y_{\mathrm{Sh}_! \, n\mathrm{Corr}^{\mathrm{enr}}(\mathcal{C})}} \mathcal{T}_n.$$

Remark 3.5. Construction 3.4 naturally breaks up into several steps. First one has $n\mathrm{Corr}^{\mathrm{enh}}(\mathcal{C})$: this is an enrichment of $n\mathrm{Corr}(\mathcal{C})$ over $\mathrm{Corr}(\mathcal{C})$, described informally by the requirement that the Hom object between a pair of $(n-1)$-cells corresponding to objects Y and Z in some overcategory $\mathcal{C}_{/S}$ is given by $Y \times_S Z$.

Then $\mathrm{Sh}_! \, n\mathrm{Corr}^{\mathrm{enh}}(\mathcal{C})$ is defined, which forms the target of the functor $n\mathrm{Sh}^{\sharp,\mathrm{pre}}$. This satisfies all the design criteria for our theorem 1.5 except for the existence of colimits: every cell in $\mathrm{Sh}_! \, n\mathrm{Corr}^{\mathrm{enh}}(\mathcal{C})$ is geometric. The desired categorifications are finally obtained by adding colimits.

4. Descent and affineness

We close this note with a discussion of the fundamental tools for working with the categorifications of a sheaf theory. We begin with a discussion of descent.

Definition 4.1. Let $\mathcal{C}$ be a category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Let $f : X \to Y$ be a map in $\mathcal{C}$ with Čech nerve $X_\bullet$. We say that f satisfies Sh-codescent if $\mathrm{Sh}(Y)$ is the geometric realization (in Pr) of $\mathrm{Sh}(X_\bullet)$. We say that f satisfies universal Sh-codescent if every base change of f satisfies Sh-codescent.

Proposition 4.2. *Let $\mathcal{C}$ be a category with finite limits, and let* $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ *be a lax symmetric monoidal functor. Let $f : X \to Y$ be a map in $\mathcal{C}$ with Čech nerve $X_\bullet$. The following are equivalent:*

(1) *f satisfies universal* Sh*-codescent.*
(2) $2\mathrm{Sh}(Y)$ *is the totalization of* $2\mathrm{Sh}(X_\bullet)$*.*
(3) $n\mathrm{Sh}(Y)$ *is the totalization of* $n\mathrm{Sh}(X_\bullet)$ *for all $n \geq 1$.*
(4) $n\mathrm{Sh}^\sharp(Y)$ *is the totalization of* $n\mathrm{Sh}^\sharp(X_\bullet)$ *for all $n \geq 1$.*

Proof. The canonical functor $2\mathrm{Sh}(Y) \to \mathrm{Tot}\, 2\mathrm{Sh}(X_\bullet)$ admits a fully faithful left adjoint. Consequently, (2) is equivalent to the assertion that the counit of the adjunction is an isomorphism. This can be checked on the generators $\mathrm{Sh}(Y'/Y)$, where $Y' \to Y$ is a map. We thus see that (2) is equivalent to the assertion that the canonical map

$$|\,\mathrm{Sh}(X'_\bullet/Y)| \to \mathrm{Sh}(Y'/Y)$$

is an isomorphism, where $X'_\bullet$ denotes the Čech nerve of the base change $f' : X' \to Y'$ of f. The above can be checked by applying Hom from an object $\mathrm{Sh}(Z/Y)$. One thus sees that (2) is equivalent to the assertion that the base change of f to $Z \times_Y Y'$ satisfies Sh-codescent. Since $Z \to Y$ and $Y' \to Y$ are arbitrary, we deduce that (2) is equivalent to (1).

Clearly (4) implies (3), which implies (2). Suppose now that (2) holds; note that it also holds for any base change of f, given the equivalence with (1). By ambidexterity for pullback and pushforward for 2Sh one sees that for every base change $X' \to Y'$ of f with Čech nerve $X'_\bullet$ we have that $2\mathrm{Sh}(Y)$ is the geometric realization of $2\mathrm{Sh}(X'_\bullet)$. Repeating the argument for the equivalence between (1) and (2) one sees that $3\mathrm{Sh}(Y)$ is the totalization of $3\mathrm{Sh}(X_\bullet)$. Arguing inductively, we deduce that $n\mathrm{Sh}(Y)$ is the totalization of $n\mathrm{Sh}(X_\bullet)$ for all $n \geq 2$. The fact that this also holds for $n = 1$ follows by passing to endomorphisms of the unit.

We have now proven that (2) implies (3), so that (1), (2) and (3) are equivalent. It remains to show that these also imply (4). Indeed, to check that $n\mathrm{Sh}^\sharp(Y)$ is the totalization of $n\mathrm{Sh}^\sharp(X_\bullet)$ it suffices to show that this is the case after applying Hom from $n\mathrm{Sh}^\sharp(Z)$ for some Z in $\mathcal{C}$. This reduces to the fact that (3) holds for arbitrary base changes of f (since (3) has been shown to be equivalent to (1), which is stable under base change). □

Remark 4.3. Let $\mathcal{C}$ be a category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Suppose that for every finite family of objects X_i in $\mathcal{C}$ the canonical functor $\mathcal{C}_{/\amalg X_i} \to \prod \mathcal{C}_{/X_i}$ is an equivalence. Then the following are equivalent:

(1) Sh preserves coproducts
(2) $n\mathrm{Sh}$ preserves coproducts for all $n \geq 1$.
(3) $n\mathrm{Sh}^\sharp$ preserves coproducts for all $n \geq 1$.

Combined with proposition 4.2, this allows one to obtain concrete criteria for checking if $n\mathrm{Sh}$ satisfies descent with respect to a Grothendieck topology.

Example 4.4. Suppose that Sh = QCoh, defined as a sheaf theory on the category of affine schemes. Then the morphisms which admit QCoh-codescent are precisely the covers for the descendable topology of [Mat16].

Example 4.5. Suppose that Sh = IndCoh, defined as a sheaf theory on the category of schemes almost of finite presentation over a field. Then every faithfully flat morphism almost of finite presentation satisfies IndCoh-codescent. It follows that nIndCoh admits fppf descent. Similarly, one has that nIndCoh admits descent along proper surjective morphisms.

Example 4.6. Suppose that $\mathrm{Sh} = \mathrm{QCoh}$, defined as a sheaf theory on quasi-compact algebraic stacks with affine diagonal and almost of finite presentation over a field of characteristic zero. Then $n\mathrm{QCoh}$ admits fppf descent thanks to [Ste25] corollary 5.1.4 (in light of propositions 3.3.5 and 4.1.11). Consequently, one sees that $n\mathrm{QCoh}$ is Kan extended from affine schemes.

We now turn to a discussion of affineness.

Proposition 4.7. *Let $\mathcal{C}$ be a category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Let $n \geq 1$. The following are equivalent:*

(1) *The canonical functor $\mathrm{Mod}_{n\mathrm{Sh}(1_{\mathcal{C}})} \to (n+1)\mathrm{Sh}(1_{\mathcal{C}})$ is an equivalence.*
(2) *For each pair of objects X, Y of $\mathcal{C}$ we have an equivalence*

$$n\mathrm{Sh}(X) \otimes_{n\mathrm{Sh}(1_{\mathcal{C}})} n\mathrm{Sh}(Y) = n\mathrm{Sh}(X \times Y).$$

Proof. The fact that (1) implies (2) follows directly from the fact that $n\mathrm{Sh}^\sharp$ is symmetric monoidal. Suppose now that (2) holds. Note that the functor $\mathrm{Mod}_{n\mathrm{Sh}(1_{\mathcal{C}})} \to (n+1)\mathrm{Sh}(1_{\mathcal{C}})$ is fully faithful. Consequently, to show (1) it will suffice to show that for every object X in $\mathcal{C}$ the unit map

$$1_{(n+1)\mathrm{Sh}(1_{\mathcal{C}})} \otimes_{n\mathrm{Sh}(1_{\mathcal{C}})} n\mathrm{Sh}(X) \to n\mathrm{Sh}(X)^{\mathrm{enh}}$$

is an isomorphism. To check this it is enough to prove that this is the case after Hom from $n\mathrm{Sh}^{\mathrm{enh}}(Y)$ for some Y, in which case our assertion reduces to the formula from (2). $\square$

A mild variant of the proof of proposition 4.7 proves the following relative version:

Proposition 4.8. *Let $\mathcal{C}$ be a category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Let $X \to Y$ be a morphism in $\mathcal{C}$, and let $n \geq 1$. The following are equivalent:*

(1) *$(n+1)\mathrm{Sh}(X) = \mathrm{Mod}_{n\mathrm{Sh}(X/Y)}((n+1)\mathrm{Sh}(Y))$.*
(2) *For every pair of maps $Z \to X \leftarrow W$ we have an equivalence*

$$n\mathrm{Sh}(Z/Y) \otimes_{n\mathrm{Sh}(X/Y)} n\mathrm{Sh}(W/Y) = n\mathrm{Sh}(Z \times_X W/Y).$$

Corollary 4.9. *Let $\mathcal{C}$ be a category with finite limits, and let $\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathrm{Pr}$ be a lax symmetric monoidal functor. Suppose that for every pair of maps $X \to S \leftarrow Y$ in $\mathcal{C}$ the canonical functor $\mathrm{Sh}(X) \otimes_{\mathrm{Sh}(S)} \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times_S Y)$ is an equivalence. Then $(n+1)\mathrm{Sh}(X) = n\mathrm{Mod}_{\mathrm{Sh}(X)}$ for all $n \geq 1$.*

Proof. This follows from an inductive application of propositions 4.7 and 4.8. $\square$

Example 4.10. Let $\mathrm{Sh} = \mathrm{QCoh}$, defined on the category of affine schemes. Then it follows from corollary 4.9 that the categorification of quasi-coherent sheaves provided by theorem 1.5 is compatible with the one we constructed in [Ste21].

Example 4.11. Let Sh be the Betti sheaf theory, defined on locally compact Hausdorff topological spaces. Then it follows from corollary 4.9 that $(n+1)\mathrm{Sh}(X) = n\mathrm{Mod}_{\mathrm{Sh}(X)}$ for all $n \geq 1$.

We finish with the following proposition, which spells out the basic affineness properties of the ind-coherent theory.

Proposition 4.12. *Let $\mathrm{Sh} = \mathrm{IndCoh}$, defined on the category of schemes almost of finite presentation over a field k.*

(1) *Let $X \to Y$ be a closed immersion. Then we have an equivalence*

$$2\mathrm{IndCoh}(X) = \mathrm{Mod}_{\mathrm{IndCoh}(X/Y)}(2\mathrm{IndCoh}(Y)).$$

(2) *Let $X \to Y$ be an arbitrary map. Then we have an equivalence*

$$n\mathrm{IndCoh}(X) = \mathrm{Mod}_{(n-1)\mathrm{IndCoh}(X/Y)}(n\mathrm{IndCoh}(Y)).$$

for all $n \geq 3$.

(3) *We have an equivalence*

$$n\mathrm{IndCoh}(\mathrm{Spec}(k)) = \mathrm{Mod}_{(n-1)\mathrm{IndCoh}(\mathrm{Spec}(k))}$$

for all $n \geq 4$.

Proof. We begin with a proof of (1). The canonical functor

$$\mathrm{Mod}_{\mathrm{IndCoh}(X/Y)}(2\mathrm{IndCoh}(Y)) \to 2\mathrm{IndCoh}(X)$$

is fully faithful, so it will suffice to show that its image generates $2\mathrm{IndCoh}(X)$. Note that $\mathrm{IndCoh}(X \times_Y X/X)$ is an object in the image. Since the projection $X \times_Y X \to X$ is a surjective closed immersion, we have that the pushforward map $\mathrm{IndCoh}(X \times_Y X/X) \to \mathrm{IndCoh}(X/X)$ admits a monadic right adjoint. Item (1) now follows from the fact that Eilenberg-Moore objects for monads in presentable higher categories can be computed in terms of Kleisli objects (which are weighted colimits).

We note that (3) follows formally from (2), in light of proposition 4.7. Item (2) may similarly be reduced to the case $n = 3$. In this case, proposition 4.7 reduces us to showing that for every pair of maps $Z \to X \leftarrow W$ we have an equivalence

$$2\mathrm{IndCoh}(Z/Y) \otimes_{2\mathrm{IndCoh}(X/Y)} 2\mathrm{IndCoh}(W/Y) = 2\mathrm{IndCoh}(Z \times_X W/Y).$$

The left hand side may be rewritten as

$$2\mathrm{IndCoh}(Z/Y) \otimes 2\mathrm{IndCoh}(W/Y) \otimes_{2\mathrm{IndCoh}(X/Y) \otimes 2\mathrm{IndCoh}(X/Y)} 2\mathrm{IndCoh}(X/Y)$$

which is equivalent to

$$2\mathrm{IndCoh}(Z \times_Y W/Y) \otimes_{2\mathrm{IndCoh}(X \times_Y X/Y)} 2\mathrm{IndCoh}(X/Y).$$

Item (2) will now follows by applying (1) to the closed immersions given by the diagonal $X \to X \times_Y X$ and its base change to $Z \times_Y W$. ☐

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